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On the Phase Diagram of the Random Field Ising Model on the Bethe Lattice

P. M. Bleher,[1] J. Ruiz,[2] and V. A. Zagrebnov[3]

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The ferromagnetic Ising model on the Bethe lattice of degree k is considered in the presence of a dichotomous external random field $\xi_x = \pm\alpha$ and the temperature $T \geqslant 0$. We give a description of a part of the phase diagram of this model in the T–α plane, where we are able to construct limiting Gibbs states and ground states. By comparison with the model with a constant external field we show that for *all realizations* $\xi = \{\xi_x = \pm\alpha\}$ of the external random field: (i) the Gibbs state is unique for $T > T_c$ ($k \geqslant 2$ and any α) or for $\alpha > 3$ ($k = 2$ and any T); (ii) the $\pm$-phases coexist in the domain $\{T < T_c, \alpha \leqslant H^{\mathrm{F}}(T)\}$, where T_c is the critical temperature and $H^{\mathrm{F}}(T)$ is the critical external field in the ferromagnetic Ising model on the Bethe lattice with a constant external field. Then we prove that for *almost all* ξ: (iii) the $\pm$-phases coexist in a larger domain $\{T < T_c, \alpha \leqslant H^{\mathrm{F}}(T) + \varepsilon(T)\}$, where $\varepsilon(T) > 0$; and (iv) the Gibbs state is unique for $3 \geqslant \alpha \geqslant 2$ at any T. We show that the residual entropy at $T = 0$ is positive for $3 \geqslant \alpha \geqslant 2$, and we give a constructive description of ground states, by so-called dipole configurations.

KEY WORDS: Random external field; Ising model; Gibbs states; ground states; Bethe lattice; residual entropy; dipole configurations; Griffiths singularities.

1. INTRODUCTION AND DEFINITIONS

It is known that the Ising model on the Bethe lattice τ_k of degree $k \geqslant 2$, exhibits rather nontrivial behavior from the point of view of the structure

[1] Department of Mathematical Sciences, Indiana University-Purdue University at Indianapolis, Indianapolis, Indiana 46202-3216; e-mail: bleher@math.iupui.edu.

[2] Centre de Physique Théorique, CNRS, Luminy case 907, F-13288 Marseille Cedex 9, France; e-mail: ruiz@cpt.univ-mrs.fr.

[3] Centre de Physique Théorique, CNRS, Luminy case 907, F-13288 Marseille Cedex 9, France, and Département de Physique, Université de la Méditerranée (Aix-Marseille II), France; e-mail: zagrebnov@cpt.univ-mrs.fr.

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of Gibbs states [BG, BRZa]. The problem gets obviously more complicated when the system is embedded into an external random field. In this paper we will be interested in the *random field Ising model* (RFIM) in the dichotomous external random field taking values $\xi_x = \pm\alpha$, $x \in V(\tau_k)$, where $V(\tau_k)$ is the set of vertices of the Bethe lattice τ_k.

The case $k=1$ corresponds to the one-dimensional RFIM. In this case there is no phase transition for the inverse temperature $\beta = T^{-1} < \infty$ (i.e., the Gibbs state $\mu_{\beta,\xi}$ is unique for all $\beta < \infty$ and all configurations $\xi = \{\xi_x, x \in V(\tau_k)\}$). The structure of the ground states $\mu_{\infty,\xi} = \lim_{\beta\to\infty} \mu_{\beta,\xi}$ is described in our previous paper [BRZb]. The exact formula for the residual entropy $S_\infty > 0$ is derived by Derrida *et al.* [DVP] (see also [BPZ, V, PF, KM]). Some partial results for $\beta < \infty$ are obtained by Bruinsma and Aeppli (see [AB, BA]). For the case $k=2$ Bruinsma [Br] proposes some clever theoretical arguments, to describe the structure of the ground states for RFIM on the Bethe lattice, and to estimate the residual entropy at $\beta \to \infty$.

The aim of the present paper is to give a rigorous study of the phase diagram of the RFIM on the Bethe lattice for the dichotomous random external field. The rest of this section contains the main notations and definitions. Section 2 is devoted to the formulation of our main results. They are summarized in Fig. 1 and they can be formulated as follows:

• The Gibbs state is unique for all α and for *all* realizations $\xi = \{\xi_x, x \in V(\tau_k)\}$ of the external field, if $T > T_c$, where T_c is the critical temperature of the model in the absence of external field. (*This result is valid for any degree* $k \geqslant 2$).

• If $k=2$ and $\alpha > 3$, then the Gibbs state $\mu_{\beta,\xi}$ is unique for all $T>0$ and *all* ξ. In this case the ground state $\mu_{\infty,\xi} = \lim_{\beta\to\infty} \mu_{\beta,\xi}$ exists, and it is concentrated on the spin configuration that follows the sign of the external field.

• If $k=2$ and $2 \leqslant \alpha \leqslant 3$, then the Gibbs state is unique for all $T>0$ and for *almost all* realizations ξ of the external field. We show that in this case the ground state exists, and it is concentrated on the set of *dipole ground state configurations* described in detail in Section 4. In Section 5 we derive an exact formula for the corresponding residual entropy. We show that the residual entropy is constant for $2 < \alpha < 3$ and it has spikes at the endpoints $\alpha = 2$ and $\alpha = 3$ (cf. [Br]).

• In the low temperature domain, $T < T_c$, we show that for all $\alpha \leqslant H^{\mathrm{F}}(T)$, where $H^{\mathrm{F}}(T) \leqslant 1$ is the critical constant external field, and for *all* ξ, there are at least two different extreme Gibbs states, which are limiting Gibbs states $\mu^{\pm}_{\beta,\xi}$ obtained with $(+)$- and $(-)$-boundary conditions.

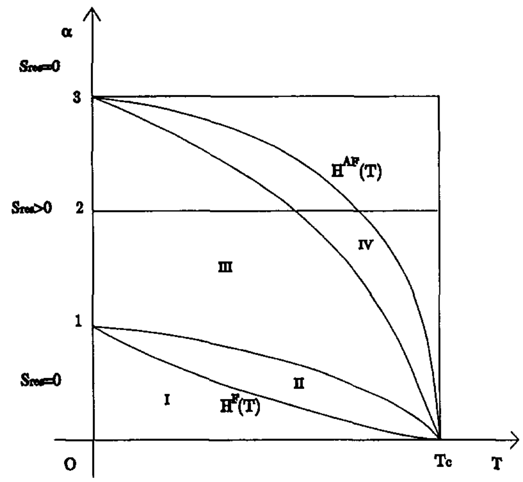

Fig. 1. Phase diagram of ferromagnetic RFIM on Bethe lattice for $k=2$. In domain *I* one has two extreme Gibbs states corresponding to $(\pm)$ boundary conditions for *all* ξ, while in domain *II* they are different for *almost all* configurations of the external field ξ. The boundary of domain *I* is the line of Giffiths singularities corresponding to a nonanalytic C^{∞} transition. Domain $\{\alpha>3$ or $T>T_c\}$ corresponds to uniqueness of the Gibbs state for *all* ξ. For $\{2\leqslant\alpha\leqslant 3$ and $T<T_c\}$ the Gibbs state is unique for *almost all* ξ and the residual entropy S_{res} is a positive constant for $2<\alpha<3$ with spikes at $\alpha=2$ and $\alpha=3$. We guess that in fact the Gibbs state is unique for *all* ξ in domain $\{\alpha>H^{AF}(T), 0<T<T_c\}$ and unique for *almost all* ξ in domain *IV*, where $H^{AF}(T)$ is again a line of Giffiths singularities. We have no guess for the rest of the domain *III*. Numerical simulations strongly suggest uniqueness for $\{1<\alpha<2, T<T_c\}$. Approximating formula for residual entropy (see Section 5) gives $S_{\mathrm{res}}>0$ for $\{1<\alpha<2, T=0\}$.

For $\alpha<1$ the corresponding ground states $\mu^{\pm}_{\infty,\xi}=\lim_{\beta\to\infty}\mu^{\pm}_{\beta,\xi}$ are concentrated on $(+)$- and $(-)$-spin configurations, respectively.

- We extend the above domain where the $(+)$- and $(-)$-Gibbs state are different, using an intermittency of the effective external field for small values of $\varepsilon=\alpha-H^{\mathrm{F}}(T)>0$ (see Section 6). Namely, we show that there exists $\varepsilon(T)$ such that in the domain $\{0<T<T_c, H^{\mathrm{F}}(T)<\alpha<H^{\mathrm{F}}(T)+\varepsilon(T)\}$, the two limiting Gibbs states with $(+)$- and $(-)$-boundary conditions are different for *almost all* realizations ξ.

The line $\alpha=H^{\mathrm{F}}(T)$ is the line of the *Griffiths singularities*. We show that on this line there is a discontinuous change of support of the probability distribution of the effective external field. We conjecture that the discontinuous change of support leads to a non-analyticity of the (internal) free energy (of, say, the $(+)$-state: on the Bethe lattice the free energy depends on the Gibbs state) as a function of α at $\alpha=H^{\mathrm{F}}(T)$. Since these averages

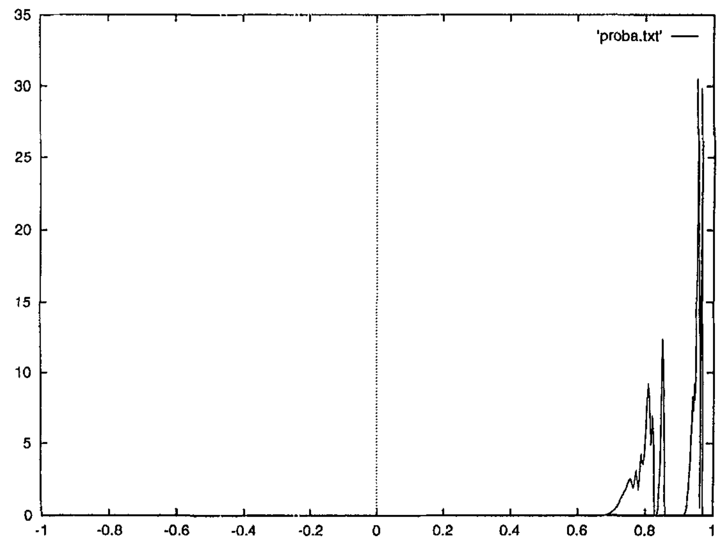

Fig. 2. Asymmetric distribution of effective field in domain *I*: $\beta = 1.0$, $\alpha = 0.4$.

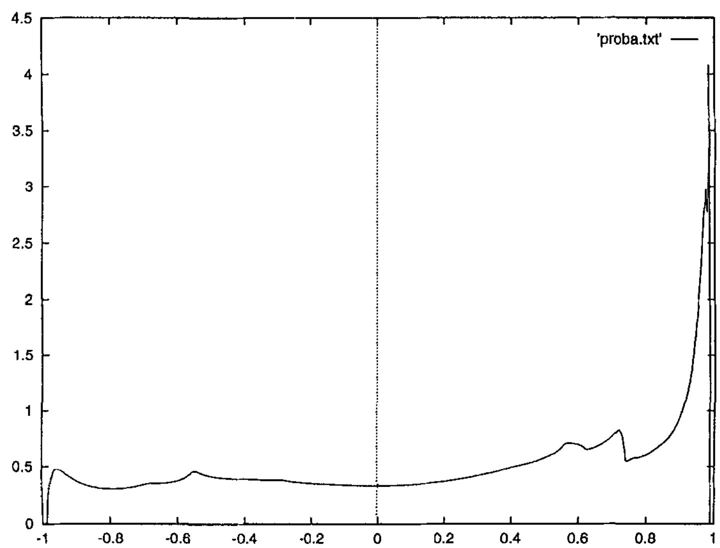

Fig. 3. Asymmetric distribution of effective field in domain *II*: $\beta = 1.0$, $\alpha = 0.8$.

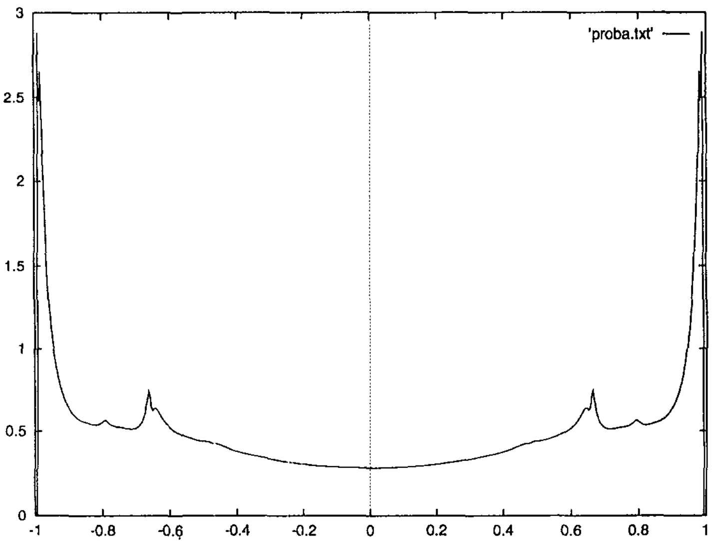

Fig. 4. Symmetric distribution of effective field in domain *III*. $\beta = 1.0$, $\alpha = 1.0$.

are C^{∞}-functions of α, the line $\alpha = H^{\mathrm{F}}(T)$ is the line of the Griffiths singularities (cf. [Br]).

• We conjecture that in the domain $\{0 < T < T_c, H^{\mathrm{AF}}(T) - \eta(T) < \alpha < H^{\mathrm{AF}}(T)\}$ the Gibbs state $\mu_{\beta, \xi}$ is unique for *almost all* ξ. The line $H^{\mathrm{AF}}(T)$ corresponds to the critical constant external field for the antiferromagnetic Ising model, see (2.1). We guess that $H^{\mathrm{AF}}(T)$ is the line of the Griffiths singularities and that the Gibbs state is unique for *all* ξ in domain $\{0 < T < T_c, H^{\mathrm{AF}}(T) < \alpha\}$.

Let us introduce the main definitions. The Hamiltonian of the ferromagnetic random field Ising model is given by

$$H = -\sum_{x, y} J_{xy} \sigma_x \sigma_y - \sum_{x} \xi_x \sigma_x \tag{1.1}$$

Here σ_x are spin variables taking values ± 1, ξ_x stands for the random external field, and $J_{xy} > 0$ are the coupling constants. We shall consider uniform interactions $J_{xy} = 1$, and a *dichotomous field*, i.e., the ξ_x are real independent random variables taking values $\pm\alpha$ with probability 1/2. These variables are defined for each site of a lattice and we shall be concerned by the *Bethe lattice* τ_k of degree $k \geqslant 2$, i.e., τ_k is a tree with exactly $k+1$ vertices coming out from each vertex x. We use V and L to denote respectively the set of vertices and edges of τ_k. There is a distance $d(x, y)$

on V which is the length of the unique path from x to y, assuming that the length of an edge is 1. Let us fix a vertex x_0 as the origin. We define

$$W_n = \{x \in V : d(x_0, x) = n\}$$

the sphere of radius n and

$$V_n = \{x \in V : d(x_0, x) \leqslant n\} = \bigcup_{m=0}^{n} W_m$$

the ball of radius n with the center at x_0. We let $L_n = \{\langle x, y\rangle : x, y \in V_n, d(x, y) = 1\}$ be the set of edges with endpoints in V_n, and for $x \in W_n$, $n = 0, 1,...,$ denote by $S(x) = \{y \in W_{n+1} : d(x, y) = 1\}$ the set of *direct sucessors* of x. Given a realization $\xi = \{\xi_x\}_{x \in V}$ of the external field, the finite-volume Gibbs measures, on the σ-algebras $\Sigma(V_n) = \{\sigma_n = \{\sigma_x = \pm 1, x \in V_n\}\}$, at inverse temperature $\beta = T^{-1}$, and boundary condition $\bar{\sigma}$ (a configuration on $V \backslash V_n$) are defined by

$$\mu_n(\sigma_n \mid \bar{\sigma}) = Z_n^{-1}(\bar{\sigma}) \exp\left\{\beta \sum_{\langle x, y\rangle \in L_n} \sigma_x \sigma_y + \beta \sum_{x \in V_n} \xi_x \sigma_x + \beta \sum_{\substack{x \in W_n \\ y \in S(x)}} \sigma_x \bar{\sigma}_y\right\} \tag{1.2}$$

where $Z_n(\bar{\sigma})$ is the partition function.

Below it will be useful to consider a more general setup of the problem corresponding to real-valued boundary conditions. It is related to construction of (nonhomogeneous) *Markov chains* on the Bethe lattice τ_k. Let $h = \{h_x, x \in V\}$ be a set of real numbers. We define for each n, the probability distributions

$$\mu_n(\sigma_n \mid h) = Z_n^{-1} \exp\left\{\beta \sum_{\langle x, y\rangle \in L_n} \sigma_x \sigma_y + \beta \sum_{x \in V_n} \xi_x \sigma_x + \beta \sum_{x \in W_n} h_x \sigma_x\right\} \tag{1.3}$$

where Z_n is the normalizing factor. These probability distributions are said to be *compatible* if for all $n > 1$,

$$\sum_{\sigma_x = \pm 1, x \in W_n} \mu_n(\sigma_n \mid h) = \mu_{n-1}(\sigma_{n-1} \mid h) \tag{1.4}$$

It is easily verified that the probability distributions (1.3) are compatible if and only if for any $x \in V$ the following equation hold:

$$h_x = \sum_{y \in S(x)} f_\beta(h_y + \xi_y) \tag{1.5}$$

where

$$f_\beta(x) = \frac{1}{2\beta} \ln \frac{\cosh \beta(x+1)}{\cosh \beta(x-1)} = \frac{1}{\beta} \operatorname{artanh}[(\tanh \beta)(\tanh \beta x)] \tag{1.6}$$

Indeed, one has

$$\sum_{\sigma_y = \pm 1} e^{\beta(\sigma_x + h_y + \xi_y)\,\sigma_y} = 2 \cosh \beta(\sigma_x + h_y + \xi_y)$$
$$= \exp\{\beta f_\beta(h_y + \xi_y)\,\sigma_x + \beta d_\beta(h_y + \xi_y)\} \tag{1.7}$$

for all $y \in W_n$. The second equality is satisfied for $\sigma_x = 1$ and $\sigma_x = -1$. This is done by taking $f_\beta(x)$ defined as above and

$$d_\beta(x) = (2\beta)^{-1} \ln[4 \cosh \beta(x+1) \cosh \beta(x-1)]$$

If the probability distributions $\mu_n(\sigma_n \mid h)$ are compatible, then by known theorems they are projections on V_n of an infinite-volume Gibbs measure $\mu(\sigma \mid h)$, and in the opposite direction, any limiting (in particular, any extreme) infinite-volume Gibbs measure $\mu(\sigma \mid h)$ has finite-volume compatible projections of the form (1.3) (cf. [G]). We will call a Gibbs state, any infinite-volume Gibbs measure $\mu(\sigma)$ on V.

By the Dobrushin–Lanford–Ruelle theorem, if $\mu(\sigma)$ is a Gibbs state then for every n, the conditional distribution of $\sigma_n = \sigma|_{V_n}$, under the condition that outside of V_n, σ coincides with some fixed configuration $\bar{\sigma}$, is given by (1.2). The conditional distributions (1.2) are called *specifications* of $\mu(\sigma)$, see e.g. [G].

Let us recall that in the case of *homogeneous* Markov chains defined by equations (1.3) when for all x, $\xi_x = H$ (constant field) and $h_x = h_*$, the equations (1.5) read

$$h_* = k f_\beta(h_* + H) \tag{1.8}$$

Then there exists $H^{\mathrm{F}}(T) \leqslant k-1$, given by the equation

$$H^{\mathrm{F}}(T) = \beta^{-1} \left[k \operatorname{artanh} \left(\frac{k\theta - 1}{k/\theta - 1} \right)^{1/2} - \operatorname{artanh} \left(\frac{k - 1/\theta}{k - \theta} \right)^{1/2} \right] \tag{1.9}$$

where $\theta = \tanh \beta$, such that the equation (1.8) has

(i) a unique solution for $T \geqslant T_c = 1/\operatorname{artanh}(1/k)$ or for $T < T_c$ and $|H| > H^{\mathrm{F}}(T)$

(ii) two distinct solutions if $T < T_c$ and $|H| = H^{\mathrm{F}}(T)$

(iii) three distinct solutions if $T < T_c$ and $|H| < H^{\mathrm{F}}(T)$

2. MAIN RESULTS

By $\mu_{\beta,\xi}$, where $0<\beta<\infty$ and $\xi=\{\xi_x=\pm\alpha, x\in V\}$, we denote a Gibbs state corresponding to specifications (1.2). The existence of $\mu_{\beta,\xi}$ for all β and configurations ξ follows from the weak compactness of the space of probability measures on $\Sigma(V)$, [G]. By ground state we understand the limit, $\mu_{\infty,\xi}=\lim_{\beta\to\infty}\mu_{\beta,\xi}$, if it exists, where $\mu_{\beta,\xi}$ is a Gibbs state.

Theorem 2.1. If $T>T_c$, then $\mu_{\beta,\xi}$ is unique for all ξ.

Theorem 2.2. Let $k=2$. If $\alpha>3$, then

(a) $\mu_{\beta,\xi}$ is unique for all $0<\beta<\infty$ and all ξ

(b) the limit (ground state) $\mu_{\infty,\xi}\equiv\lim_{\beta\to\infty}\mu_{\beta,\xi}$ exists and it is concentrated on a configuration such that $\sigma_x=\text{sign }\xi_x$ for all $x\in V$.

If $3\geqslant\alpha\geqslant 2$, then $\mu_{\beta,\xi}$ is unique for all ξ and all $0<\beta<\infty$ such that

$$\tanh[\beta(\alpha-1)]+\tanh[\beta(3-\alpha)]\leqslant 1$$

Remark. Observe that for $\alpha=2$ the last condition reduces to $2\tanh\beta\leqslant 1$, which is equivalent to $\beta\leqslant\beta_c$. The domain of "the uniqueness for all ξ" in Theorems 2.1 and 2.2 can be probably extended as follows. Consider the critical constant external field of the *antiferromagnetic* Ising model on the Bethe lattice:

$$H^{\mathrm{AF}}(T)=\beta^{-1}\left[k\,\operatorname{artanh}\left(\frac{k\theta-1}{k/\theta-1}\right)^{1/2}+\operatorname{artanh}\left(\frac{k-1/\theta}{k-\theta}\right)^{1/2}\right]\leqslant k+1 \tag{2.1}$$

where $\theta=\tanh\beta$ (see Fig. 1). We conjecture that if $T<T_c$ and $\alpha>H^{\mathrm{AF}}(T)$, then the Gibbs state $\mu_{\beta,\xi}$ is unique for *all* ξ (see a discussion in the proof of Theorem 2.2 below). It is also plausible that there exists a continuous function $\eta(T)>0$ on $0<T<T_c$ such that if $T<T_c$ and $H^{\mathrm{AF}}(T)>\alpha>H^{\mathrm{AF}}(T)-\eta(T)$, then the Gibbs state $\mu_{\beta,\xi}$ is unique for *almost all* ξ. The critical line $\alpha=H^{\mathrm{AF}}(T)$ is probably the line of the Griffiths singularities (cf. [Br]), like the ferromagnetic critical line $\alpha=H^{\mathrm{F}}(T)$. In other words, we expect that some of thermodynamic averages are C^∞ non-analytic functions of α at $\alpha=H^{\mathrm{AF}}(T)$.

Theorem 2.3. Assume that $0<T<T_c$ and $\alpha\leqslant H^{\mathrm{F}}(T)$, see (1.9). Then for any $k\geqslant 2$ and all realizations ξ of the external field,

(a) there exist two different extreme Gibbs states $\mu^+_{\beta,\xi}$ and $\mu^-_{\beta,\xi}$ which are limiting Gibbs states with $+$ and $-$ boundary conditions;

(b) if $\alpha < 1$ then the limits (ground states) $\mu^{\pm}_{\infty,\xi} \equiv \lim_{\beta\to\infty} \mu^{\pm}_{\beta,\xi}$ exist and they are concentrated on configurations $\{\sigma_x = 1, x \in V\}$ and $\{\sigma_x = -1, x \in V\}$, respectively.

Theorem 2.4. Let $k = 2$ and assume that $2 \leqslant \alpha \leqslant 3$. Then

(a) for all $\beta < \infty$ and for almost all realizations ξ of the external field, there exists a unique Gibbs state $\mu_{\beta,\xi}$

(b) for almost all ξ, there exists a ground state $\mu_{\infty,\xi} \equiv \lim_{\beta\to\infty} \mu_{\beta,\xi}$ and it is a probability measure concentrated on the set of dipole ground state configurations which are described in Section 4 below.

The residual entropy S_∞ of the ground state $\mu_{\infty,\xi}$ is calculated in Section 5.

Theorem 2.5. Let $k = 2$. Then there exists a positive continuous function $\varepsilon(T)$ on $0 \leqslant T \leqslant T_c$ ($\varepsilon(0) = \varepsilon(T_c) = 0$) such that if $0 < T < T_c$ and $H^{\mathrm{F}}(T) < \alpha < H^{\mathrm{F}}(T) + \varepsilon(T)$, then for almost all ξ, $\mu^{+}_{\beta,\xi} \neq \mu^{-}_{\beta,\xi}$.

Theorem 2.5 is derived from the following

Theorem 2.6. Under the assumptions of Theorem 2.5, let $0 < T < T_c$ and $H^{\mathrm{F}}(T) < \alpha < H^{\mathrm{F}}(T) + \varepsilon(T)$. Then under the assumptions of Theorem 2.5 one has:

$$\mathbb{E}_\xi \int \sigma_x \mu^{+}_{\beta,\xi}(\sigma) \equiv \mathbb{E}_\xi \langle \sigma_x \rangle^{+}_{\beta\xi} > 0$$

Before passing to proofs we would like to make few remarks about our results. Theorems 2.1, 2.2, and 2.3 are valid for *all* realizations ξ of the random external field and these theorems are relatively easy. To prove them we use some contraction estimates and F.K.G. correlation inequalities to show that the ferromagnetic Ising model on the Bethe lattice with a *dichotomous random* external field is majorized, in an appropriate sense, by the model with a *constant* external field of the same strength.

On the contrary, Theorems 2.4 and 2.5 are valid only for *almost all* realizations of ξ and their proof is much more difficult. It is worth to notice that in these theorems the condition "for almost all realizations ξ" cannot be replaced by the one "for all realizations ξ." For instance, in Theorem 2.4 one can take $2 \leqslant \alpha < H^{\mathrm{AF}}(T)$ and a *chess-board* realization ξ (a realization with alternating pluses and minuses). Then the Gibbs state $\mu_{\beta,\xi}$ is *not* unique for this ξ, although by Theorem 2.4 it is *unique* for *almost all* ξ. Similarly, in Theorem 2.5, $\mu^{+}_{\beta,\xi} = \mu^{-}_{\beta,\xi}$ if one takes $\xi_x \equiv \alpha > H^{\mathrm{F}}(T)$, a *constant realization*, while $\mu^{+}_{\beta,\xi} \neq \mu^{-}_{\beta,\xi}$ for *almost all* ξ.

An interesting feature of the Ising model with the dichotomous random external field is that the residual entropy S_∞ at $T=0$ is *positive* for $2 \leqslant \alpha \leqslant 3$. We conjecture that it is *positive* in the interval $1 \leqslant \alpha < 2$ as well but we cannot prove it. Convincing heuristic arguments in favor of this conjecture are given by Bruinsma [Br]. We calculate S_∞ in Section 5 below, and it turns out that $S_\infty(\alpha) = const$ in the interval $2 < \alpha < 3$ while at $\alpha = 2$ and $\alpha = 3$ the residual entropy has two *spikes*. This behavior of the residual entropy is easily explained by the structure of the Gibbs measures in the limit $T \to 0$. Namely, for all $2 < \alpha < 3$ the limiting Gibbs measure $\mu_{\infty, \xi} = \lim_{\beta \to \infty} \mu_{\beta, \xi}$ is *independent* of α, while for $\alpha = 2$ and $\alpha = 3$ it is concentrated on much bigger sets of configurations than for $2 < \alpha < 3$ (see Section 4). Bruinsma [Br] derives a good approximate formula for $S_\infty(\alpha)$ and he shows that this approximate formula predicts that $S_\infty(\alpha)$ is constant in every interval $1 + (2/n+1) < \alpha < 1 + 2/n$, $n = 1, 2,...,$ with spikes at $\alpha = 1 + 2/n$.

The central point in the proof of Theorem 2.5 in Section 6 is to show that the limiting probability distribution $\nu(dh_x)$ of the effective external field h_x is not symmetric under plus boundary conditions. To prove the asymmetry of $\nu(dh_x)$ we use the intermittency of the iterations of h_x for a small difference $\alpha - H^{\mathrm{F}}(T) > 0$, and we show that the main mass of $\nu(dh_x)$ is concentrated on the positive half-axis, which gives Theorem 2.6. Then we derive Theorem 2.5 from Theorem 2.6 using some soft ergodic arguments.

3. PROOF OF THEOREMS 2.1–2.3 AND 2.4a

We introduce the variables $g_x = f_\beta(\xi_x + h_x)$. Then the recursive equation (1.5) reads

$$h_x = \sum_{y \in S(x)} g_y \tag{3.1}$$

This implies that g_x satisfies

$$g_x = f_\beta \left(\xi_x + \sum_{y \in S(x)} g_y \right) \tag{3.2}$$

By (1.3) and (3.1), the probability distribution $\mu(\sigma_n \mid g)$, $\sigma_n \equiv \sigma_{|V_n}$, can be written as

$$\mu(\sigma_n \mid g) = Z_n^{-1} \exp \left\{ \beta \sum_{\langle x, y \rangle \in L_n} \sigma_x \sigma_y + \beta \sum_{x \in V_n} \xi_x \sigma_x + \beta \sum_{\substack{x \in W_n \\ y \in S(x)}} \sigma_x g_y \right\} \tag{3.3}$$

We freely call $g=\{g_x\}$ the *effective* external field, along with $h=\{h_x\}$. We recall that by F.K.G. inequalities [FKG] one has the following proposition (see [LM]).

Proposition 3.1. The Gibbs states $\mu^+_{\beta,\xi}$ and $\mu^-_{\beta,\xi}$ exist and they are extreme for all ξ. If $\mu^+_{\beta,\xi}=\mu^-_{\beta,\xi}$, for a given ξ, then the Gibbs state is unique for this ξ.

We denote by $g^\pm=\{g^\pm_x, x\in V\}$ the configurations that correspond to the Gibbs states $\mu^\pm_{\beta,\xi}$.

We use below some properties of the function (1.6). They are summarized in the next

Proposition 3.2.

$$f_\beta(-x)=-f_\beta(x), \qquad f_\beta(\infty)=1$$

$$0<\frac{d}{dx}f_\beta(x)<\tanh\beta \quad \forall x\neq 0, \qquad \frac{d}{dx}f_\beta(0)=\tanh\beta \tag{3.4}$$

$$\frac{d}{dx}f_\beta(x)<\frac{1}{2} \qquad \text{if} \quad x\geqslant 1 \tag{3.5}$$

$$\frac{d}{dx}f_\beta(x)<\frac{1}{2}(1-\tanh\beta) \qquad \text{if} \quad x\geqslant 2 \tag{3.6}$$

$$\frac{d^2}{dx^2}f_\beta(x)<0 \quad \forall x>0, \qquad \frac{d^2}{dx^2}f_B(0)=0 \tag{3.7}$$

Proof. All the relations (3.4)–(3.7) result from the following two equations:

$$\begin{aligned}\frac{d}{dx}f_\beta(x)&=\frac{1}{2}[\tanh\beta(x+1)-\tanh\beta(x-1)]\\ \frac{d^2}{dx^2}f_\beta(x)&=\frac{\beta}{2}\left[\frac{1}{\cosh^2\beta(x+1)}-\frac{1}{\cosh^2\beta(x-1)}\right] \quad \blacksquare\end{aligned} \tag{3.8}$$

Now we turn to the proof of Theorems 2.1–2.3 and 2.4a.

Proof of Theorems 2.1 and 2.2. Consider the set of recursive equations (3.2). We have

$$g^+_x-g^-_x=f'_\beta(c)\sum_{y\in S(x)}(g^+_y-g^-_y)$$

for some $c \in [\xi_x + \sum_{y \in S(x)} g_y^-, \xi_x + \sum_{y \in S(x)} g_y^+]$ so that

$$|g_x^+ - g_x^-| \leqslant k\, |f'_\beta(c)| \sup_{y \in S(x)} |g_y^+ - g_y^-| \tag{3.9}$$

When $T > T_c$, one has $\tanh \beta < 1/k$, so that by (3.4), $f'_\beta(c) < 1/k$. By applying recursively the inequality (3.9), we get that $g_x^+ = g_x^-$. Hence $\mu^+_{\beta,\xi} = \mu^-_{\beta,\xi}$. In virtue of Proposition 3.1 this proves Theorem 2.1.

To prove Theorem 2.2, we first remark that by Proposition 3.2, $|f_\beta(t)| < 1$. This implies that $|g_y| < 1 - \delta$ for some $\delta = \delta(\alpha, \beta, k) > 0$ and all y. Hence, when $\alpha \geqslant 2$ and $k = 2$, we have that $|c| > \alpha - 2 + 2\delta$ in (3.9). By (3.8),

$$f'_\beta(c) = \tfrac{1}{2}\{\tanh \beta(|c| + 1) - \tanh \beta(|c| - 1)\}$$

hence for some $\delta_0 > 0$,

$$0 < f'_\beta(c) < \tfrac{1}{2} - \delta_0$$

provided $\alpha \geqslant 3$ or $3 > \alpha \geqslant 2$ and

$$\tanh \beta(\alpha - 1) + \tanh \beta(3 - \alpha) \leqslant 1$$

In the both cases, iterating (3.9) we conclude that $g_x^+ = g_x^-$. Hence, the uniqueness part of Theorem 2.2 follows from Proposition 3.1.

It is interesting to notice that the worst estimate on $f'(c)$ occurs when the quantity $|\xi_x + g_y + g_z|$ is minimal. For $\alpha \geqslant 2$ this happens when the sign of ξ_x is opposite to the sign of ξ_y and ξ_z, or if we extend this property to the whole lattice, when ξ is a chessboard configuration. The chessboard ξ is equivalent (by a gauge transformation) to the antiferromagnetic model with constant magnetic field, and this motivates our conjecture that the uniqueness *for all* ξ holds for $\alpha > H^{\mathrm{AF}}(T)$.

For the Statement (b) of Theorem 2.2, we observe that for $n = 1$ one gets by (3.3) that

$$\mu(\sigma_x \mid g) = Z^{-1} \exp\left\{\beta\left[\xi_x + \sum_{y\,:\,d(x,y)=1} g_y\right]\sigma_x\right\}$$

When $|\xi_x| > 3$ then $|\xi_x + \sum_{y\,:\,d(x,y)=1} g_y| > 0$ and $\mathrm{sign}(\xi_x + \sum_{y\,:\,d(x,y)=1} g_y) = \mathrm{sign}\, \xi_x$. Thus we finish the proof by taking the limit $\beta \to \infty$. ∎

Proof of Theorem 2.3. Let $\langle\cdot\rangle^{\pm}_{\beta\xi}$ denote the expectation with respect to the measure $\mu^{\pm}_{\beta,\xi}$. Then by the conditions of Theorem 2.3 and by the F.K.G. inequality one gets (cf. the one-point measure above) that

$$\langle\sigma_x\rangle^{+}_{\beta\xi} \geqslant \langle\sigma_x\rangle^{+}_{\beta\{-\alpha\}} > 0$$

and

$$\langle\sigma_x\rangle^{-}_{\beta\xi} \leqslant \langle\sigma_x\rangle^{-}_{\beta\{\alpha\}} < 0$$

This proves that $\mu^{+}_{\beta,\xi} \neq \mu^{-}_{\beta,\xi}$. Their extremality follows from Proposition 3.1. Since for $\alpha < 1$ $\lim_{\beta\to\infty}\langle\sigma_x\rangle^{\pm}_{\beta\{\mp\alpha\}} = \pm 1$, we obtain that for all realizations ξ, $\sigma_x = 1$ a.e. with respect to $\mu^{+}_{\beta,\xi}$ and $\sigma_x = -1$ a.e. with respect to $\mu^{-}_{\beta,\xi}$. ∎

Proof of Statement (a) Theorem 2.4. For any x, we denote by y and z, its two direct successors. The recursive equation (3.2) reads

$$g_x = f_\beta(\xi_x + g_y + g_z) \tag{3.10}$$

For a given ξ, let g_x^+ and g_x^- be the g_x corresponding respectively to the states $\mu^{+}_{\beta,\xi}$ and $\mu^{-}_{\beta,\xi}$. We shall estimate recursively the expectation $\mathbb{E}_\xi\ |g_x^+ - g_x^-|$. We have

$$\begin{aligned} g_x^+ - g_x^- &= f_\beta(\xi_x + g_y^+ + g_z^+) - f_\beta(\xi_x + g_y^- + g_z^-) \\ &= f'_\beta(c)[g_y^+ - g_y^- + g_z^+ - g_z^-] \end{aligned} \tag{3.1}$$

where $c \in [\xi_x + g_y^- + g_z^-, \xi_x + g_y^+ + g_z^+]$. Let us estimate $f'_\beta(c)$. Assume that $\xi_x = \alpha > 0$. Consider different cases for ξ_y and ξ_z.

Case (i) $\xi_y = \xi_z = -\alpha$. Then we use the estimate

$$f'_\beta(c) \leqslant \tanh\beta \tag{3.12}$$

which is valid for all c (see (3.4))

Case (ii) $\xi_y + \xi_z = 0$. Let for instance $\xi_y = \alpha$ and $\xi_z = -\alpha$. Then

$$g_y^\pm > 0, \qquad g_z^\pm > -1$$

hence

$$\xi_x + g_y^\pm + g_z^\pm > 1, \qquad c > 1$$

In this case

$$f'_\beta(c) < \tfrac{1}{2} \tag{3.13}$$

(see (3.5)).

Case (iii) $\xi_y = \xi_z = \alpha$. Then $\xi_x + g_y^\pm + g_z^\pm > 2$, hence $c > 2$ and

$$f'_\beta(c) < \tfrac{1}{2}(1 - \tanh\beta) \tag{3.14}$$

(see (3.6)).

Notice that the probabilities of the cases (i), (ii) and (iii) are, respectively, 1/4, 1/2, and 1/4. Thus by (3.11)–(3.14) one gets for $x \in W_n$ that

$$\mathbb{E}_\xi\, |g_x^+ - g_x^-| < [\tfrac{1}{4}\cdot\tanh\beta + \tfrac{1}{2}\cdot\tfrac{1}{2} + \tfrac{1}{4}\cdot\tfrac{1}{2}\cdot(1 - \tanh\beta)]\cdot 2\cdot E_{n+1}$$

where

$$E_{n+1} = \max_{t \in W_{n+1}} \mathbb{E}_\xi\, |g_t^+ - g_t^-|$$

This gives that

$$E_n < \frac{3 + \tanh\beta}{4} E_{n+1}$$

Since $(3 + \tanh\beta)/4 < 1$ and $E_n < 2$ for all n, this implies by iterations that $E_n = 0$ for all n. Hence for all x, $g_x^+ = g_x^-$, for almost all configurations ξ, which implies uniqueness by Proposition 3.1. ▌

4. DIPOLE GROUND STATES

Assume that $k = 2$ and $\xi_x = \pm\alpha$ with $2 < \alpha < 3$. We discuss the ground state, $\mu_{\infty,\xi}(\sigma) = \lim_{\beta\to\infty} \mu_{\beta,\xi}(\sigma)$. Let $x \in V$. Then the one-site projection of the Gibbs measure (cf. Section 3) can be presented as

$$\mu_{\beta,\xi}(\sigma_x) = Z^{-1} \exp[\beta\sigma_x(\xi_x + g_{yx} + g_{zx} + g_{tx})] \tag{4.1}$$

where y, z, t are the nearest neighbors of x. In this section we do not fix the origin x_0 in V, and it is more convenient for us to consider the effective field as a function on oriented edges, $g = \{g_{xy}\}$, rather than a function on vertices $g = \{g_x\}$. In this context, what was before g_x, $x \neq x_0$ is now denoted by g_{xy} where the edge $\langle x, y\rangle$ goes from x in the direction of the origin x_0. The numbers g_{xy} satisfy the recursive equation

$$g_{xy} = f_\beta(\xi_x + g_{zx} + g_{tx}) \tag{4.2}$$

At $\beta = \infty$

$$g_{xy} = (\xi_x + g_{zx} + g_{tx}) \tag{4.3}$$

where $f_\infty = \lim_{\beta \to \infty} f_\beta$ is the piecewise linear function:

$$f_\infty(x) = \begin{cases} -1 & \text{if } x \leqslant -1 \\ x & \text{if } -1 \leqslant x \leqslant 1 \\ 1 & \text{if } x \geqslant 1 \end{cases} \tag{4.4}$$

For $2 < \alpha < 3$ and almost all ξ, the equation (4.3) has a unique solution $g_\xi = \{g_{xy}(\xi)\}$ with

$$g_{xy}(\xi) \in \{-1, -\varepsilon, \varepsilon, 1\} \tag{4.5}$$

where

$$\alpha = 2 + \varepsilon, \qquad 0 < \varepsilon < 1 \tag{4.6}$$

(see Lemma 4.2 below). Consider the partition of V in three subsets, $V = V_+ \cup V_- \cup V^0$ with

$$V_\pm = \{x \in V : \pm(\xi_x + g_{yx} + g_{zx} + g_{tx}) > 0\}$$
$$V^0 = \{x \in V : \xi_x + g_{yx} + g_{zx} + g_{tx} = 0\}$$

where $g = \{g_{xy}\}$ is the unique solution of (4.3), $g = g(\xi)$. The sets $V_\pm$, V^0 depend on ξ. We show below that for almost all ξ the equation (4.2) has a unique solution $g = g(\xi, \beta)$ and the limit $\lim_{\beta \to \infty} g(\xi, \beta) = g(\xi, \infty)$, exists. Then $g(\xi, \infty)$ is the unique solution to (4.3). From (4.1) it is clear that

$$\mu_{\infty, \xi}(\sigma_x = 1) = \begin{cases} 1 & \text{if } x \in V_+ \\ 0 & \text{if } x \in V_- \end{cases} \tag{4.7}$$

In other words, at $\beta = \infty$, $\sigma_x = 1$ on V_+ and $\sigma_x = -1$ on V_-.

Let us describe σ_x on V^0. Let $x \in V^0$. Assume, for the sake of definiteness, that $\xi_x = 2 + \varepsilon$. Then

$$2 + \varepsilon + g_{yx} + g_{zx} + g_{tx} = 0$$

and all g's are from the set $\{\pm 1, \pm \varepsilon\}$. Hence two of them, say g_{yx} and g_{zx}, are -1 and $g_{tx} = -\varepsilon$ (see Fig. 5).

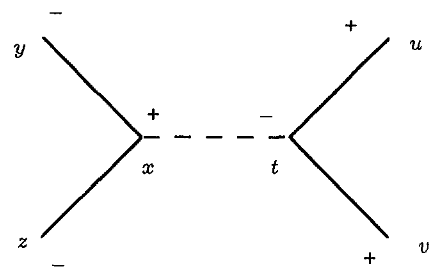

Fig. 5. Ground state of dipole $\{x, t\}$ configuration with $x, t \in V_0$. The signs $(+, -)$ correspond to distribution of the external field (charges). "Charge" $\xi_x = 2 + \varepsilon$ polarizes the neighbours to take opposite signs. The same for the charge $\xi_t = -(2+\varepsilon)$. The dashed line corresponds to the "current" $0 < g_{xt} = -g_{tx} = \varepsilon < 1$, while solid lines correspond to "currents" $= \pm 1$. Positive currents are directed from positive to negative charges.

Then by (4.3),

$$-\varepsilon = g_{tx} = \xi_t + g_{ut} + g_{vt}$$

which implies that

$$\xi_t = -2 - \varepsilon, \qquad g_{ut} + g_{vt} = 2 \tag{4.8}$$

By (4.3),

$$\begin{aligned} g_{xt} &= \xi_x + g_{yx} + g_{zx} = 2 + \varepsilon - 1 - 1 = \varepsilon \\ g_{tu} &= \xi_t + g_{xt} + g_{vt} = -2 - \varepsilon + \varepsilon - 1 = -1 \\ g_{tv} &= \xi_t + g_{xt} + g_{ut} = -1 \end{aligned} \tag{4.9}$$

Thus,

$$\xi_t + g_{xt} + g_{ut} + g_{vt} = -2 - \varepsilon + \varepsilon + 1 + 1 = 0$$

so that $t \in V^0$.

This proves that if $x \in V_0$ and $\xi_x = 2 + \varepsilon$ then there is a neighboring vertex t such that $t \in V^0$, $\xi_t = -2 - \varepsilon$, and $g_{tx} = -g_{xt} = \varepsilon$. This motivates the following

Definition 4.1. Two neighboring vertices x,t are called a *dipole* if

$$x, t \in V^0, \qquad \xi_x + \xi_t = 0, \qquad |g_{xt}| = |g_{tx}| = \varepsilon \tag{4.10}$$

We call ξ_x a *charge* at x and g_{xt} a *current* from x to t. We call any connected component of the set V^0 a *dipole polymer*, where we assume that two vertices $x, t \in V^0$ are connected if $d(x, t) = 1$.

Observe that in any dipole $\{x, t\}$, the charges ξ_x and ξ_t have opposite signs and the currents are

$$-g_{px} = g_{xp} = \begin{cases} \varepsilon \operatorname{sign}(\xi_x) & \text{if} \quad p = t \\ \operatorname{sign} \xi_x & \text{if} \quad p \neq t \end{cases} \tag{4.11}$$

In addition,

$$\xi_y = \xi_z = \xi_t, \qquad \xi_u = \xi_v = \xi_x \tag{4.12}$$

so that the charges in the dipole attract from the outside the charges of the opposite sign (see Fig. 6). Indeed, consider, for instance, ξ_u. From (4.8) and (4.3), $1 = g_{ut} = \xi_u + g_{pu} + g_{qu}$, where p and q are nearest neighbors of u, which implies that ξ_u cannot be $-2-\varepsilon$, hence $\xi_u = 2 + \varepsilon$. This proves (4.12).

Every *dipole polymer* consists of dipoles, as shown in Fig. 6. In Fig. 6 the dipole bonds are shown by dash lines, and the bonds connecting dipoles between themselves and with the environment are shown by solid lines. Observe that $g_{xy} = -g_{yx} = \pm\varepsilon$ on dash lines and $g_{xy} = -g_{yx} = \pm 1$ on solid lines. The sign of g_{xy} is determined by the rule that positive current goes from $+$ to $-$.

In any dipole polymer the charges are alternating. This implies that for almost all ξ there is no infinite polymer. On the other hand, there is a

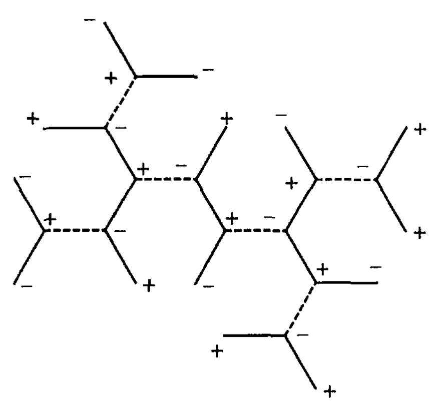

Fig. 6. Configuration of "dipole polymer" corresponding to a ground-state configuration of charges $\{\xi_x = \pm(2-\varepsilon)\}$ and "currents": $\{g_{xy} = \pm\varepsilon\}$ (dashed lines) and $\{g_{uv} = \pm 1\}$ (solid lines).

positive probability of appearing a given polymer at a given place in V. Hence V^0 consists of an infinite number of finite dipole polymers, $V^0 = \bigcup_{k=1}^{\infty} V_k$, and the dipole polymers V_k have a positive density on V.

Consider a polymer V_k. Assume that $x \in V_k$ and $y \notin V_k$ with $d(x, y) = 1$. Then

$$y \in V_\tau, \qquad \tau = \text{sign } \xi_y \tag{4.13}$$

Indeed, let for the sake of definiteness $\xi_y = 2 + \varepsilon$. Then

$$1 = g_{yx} = f_\infty(\xi_y + g_{py} + g_{qy}) = f_\infty(2 + \varepsilon + g_{py} + g_{qy})$$

which shows that $g_{py} + g_{qy} \geqslant -(1 + \varepsilon)$. Hence

$$\xi_y + g_{xy} + g_{py} + g_{qy} \geqslant 2 + \varepsilon - 1 - 1 - \varepsilon = 0$$

so that y cannot be from V_-. Since $y \notin V^0$, this implies that $y \in V_+$. (4.13) is proved. Thus on the boundary of a polymer V_k the charge of any boundary point y determines the component $V_\pm$ to which y belongs.

Let $\{x, t\}$ be a dipole. Let us determine the possible values σ_x, σ_t of a ground state. Assume first that $2 < \alpha < 3$. We have that

$$\mu_{\beta,\xi}(\sigma_x, \sigma_t) = Z^{-1} \exp\{\beta[\sigma_x\sigma_t + (\xi_x + g_{yx} + g_{zx})\,\sigma_x + (\xi_t + g_{ut} + g_{vt})\,\sigma_t]\}$$

Since $x, t \in V^0$,

$$\xi_x + g_{yx} + g_{zx} = -g_{tx}, \qquad \xi_t + g_{ut} + g_{vt} = -g_{xt}$$

and

$$\mu_{\beta,\xi}(\sigma_x, \sigma_t) = Z^{-1} \exp\{\beta(\sigma_x\sigma_t + g_{xt}\sigma_x + g_{tx}\sigma_t)\} \tag{4.14}$$

Assume, for the sake of definiteness, that $\xi_x = 2 + \varepsilon$, $\xi_t = -2 - \varepsilon$. Then $g_{tx} = -\varepsilon$, $g_{xt} = \varepsilon$ and

$$\mu_{\beta,\xi}(\sigma_x, \sigma_t) = Z^{-1}\{\exp \beta(\sigma_x\sigma_t + \varepsilon\sigma_x - \varepsilon\sigma_t)\} \tag{4.15}$$

Since $0 < \varepsilon < 1$, the ground states $(\beta \to \infty)$ correspond to $\sigma_x\sigma_t = 1$, i.e., $\sigma_x = \sigma_t = 1$, $\sigma_x = \sigma_t = -1$. In other words, on any dipole we have either the $(+)$-state or the $(-)$-state.

Let $x, t \in V_k$ be nearest neighbors belonging to two different dipoles. Assume that $\xi_x = 2 + \varepsilon$, $\xi_t = -2 - \varepsilon$. Then $g_{tx} = -1$, $g_{xt} = 1$ and (4.14) reduces to

$$\mu_{\beta,\xi}(\sigma_x, \sigma_t) = Z^{-1} \exp \beta(\sigma_x\sigma_t + \sigma_x - \sigma_t) \tag{4.16}$$

and the ground states are

$$(\sigma_x = 1, \sigma_t = 1), (\sigma_x = -1, \sigma_t = -1), (\sigma_x = 1, \sigma_t = -1) \qquad (4.17)$$

In other words, between two dipoles we can change the sign of σ_x from sign ξ_x to sign ξ_t.

Definition 4.2. Assume that $2 < \alpha < 3$. Let $V_k = V_k(\xi)$ a dipole polymer. A configuration $\sigma = \{\sigma_x, x \in V_k\}$ on V_k is called a *dipole ground state configuration* if

(i) $\sigma_x \sigma_t = 1$ for every dipole $\{x, t\}$ in V_k,

(ii) either $\sigma_x \sigma_t = 1$ or $\{\sigma_x = \text{sign } \xi_x, \sigma_t = \text{sign } \xi_t\}$ for every pair $\{x, t\}$ connecting two dipoles in V_k.

We denote by $M_k = M_k(\xi)$ the set of dipole ground state configurations on $V_k(\xi)$.

Definition 4.2 describes a dipole ground state configuration for the case when $2 < \alpha < 3$. For the cases $\alpha = 3$ and $\alpha = 2$ it should be modified as follows. Observe that for $\varepsilon = 1$ $(\alpha = 3)$ formula (4.15) coincides with (4.16), hence the ground states on a dipole are (4.17). This leads to the following

Definition 4.2, for $\alpha = 3$. Let $V_k = V_k(\xi)$ be a dipole polymer. A configuration $\sigma = \{\sigma_x, x \in V_k\}$ on V_k is called a dipole ground state configuration if for every neighboring $x, t \in V_k$ either $\sigma_x \sigma_t = 1$ or $\{\sigma_x = \text{sign } \xi_x, \sigma_t = \text{sign } \xi_t\}$.

Notice that the difference between a ground state configuration for $2 < \alpha < 3$ and for $\alpha = 3$ is that for $2 < \alpha < 3$, $\sigma_x \sigma_t = 1$ on any dipole while for $\alpha = 3$ either $\sigma_x \sigma_t = 1$ or $\{\sigma_x = \text{sign } \xi_x, \sigma_t = \text{sign } \sigma_t\}$. Therefore the number of ground state configurations for $\alpha = 3$ is bigger than the number of those for $2 < \alpha < 3$. This is reflected in the behavior of the residual entropy, which is higher at $\alpha = 3$ (at the spike) than at $2 < \alpha < 3$ (on the plateau). We evaluate the residual entropy in the next section.

When $\alpha = 2$ we have to change the definition of dipole.

Namely, in the case $\alpha = 2$, if $x \in V^0$ and, and, say, $\xi_x = 2$, $g_{yx} = -1$, $g_{zx} = -1$, $g_{tx} = 0$, then $t \in V^0$ but ξ_t can be both 2 and -2. Thus we arrive at the following

Definition 4.1, for $\alpha = 2$. Two neighboring vertices x, t are a dipole if

$$x, t \in V^0, \qquad g_{xt} = g_{tx} = 0$$

Thus, we do not have the restriction $\xi_x + \xi_t = 0$ as in (4.10). Strictly speaking, (ξ_x, ξ_t) is not a dipole anymore, since it is not necessarily neutral. We will call (x, t) a dipole to facilitate a unique formulation of a ground state both for $2 < \alpha \leqslant 3$ and for $\alpha = 2$. The definition of a dipole ground state configuration for $\alpha = 2$ remains the same as in Definition 4.2. Observe that for $\alpha = 2$ we have more dipoles and, consequently, more dipole polymers than for $2 < \alpha < 3$. This produces a jump of the residual entropy for $\alpha = 2$ (see Section 5).

Theorem 4.1. Let $k = 2$ and $2 \leqslant \alpha \leqslant 3$. Then for almost all ξ there exist a limit, $\mu_{\infty, \xi}(\sigma) = \lim_{\beta \to \infty} \mu_{\beta, \xi}(\sigma)$ and

$$\mu_{\infty, \xi}(\sigma) = \mu_\infty^+(\sigma_{V_+})\, \mu_\infty^-(\sigma_{V_-}) \prod_{k=1}^{\infty} \mu_{\infty, \xi}^k(\sigma_{V_k})$$

where $\mu_\infty^\pm(\sigma_{V_\pm})$ is a degenerate measure concentrated on $\{\sigma_x = +1\}$ or $\{\sigma_x = -1\}$, respectively, and $\mu_{\infty, \xi}^k(\sigma_{V_k})$ is a uniform measure on the set M_k of dipole ground state configurations, so that $\mu_{\infty, \xi}^k(\sigma_{V_k}) = 1/|M_k|$ for all dipole ground state configurations σ_{V_k}.

Proof of Theorem 4.1. Let $\Lambda \subset V$ be a finite subset. Then

$$\mu_{\beta, \xi}(\sigma_\Lambda) = Z^{-1} \exp \beta \left(\sum_{\substack{\langle x, y \rangle \\ x, y \in \Lambda}} \sigma_x \sigma_y + \sum_{x \in \Lambda} \xi_x \sigma_x + \sum_{\substack{\langle x, y \rangle \\ x \in \Lambda,\, y \in \Lambda^c}} g_{yx} \sigma_x \right) \tag{4.18}$$

Lemma 4.1. For every $N > 1$ and almost all ξ, as $\beta \to \infty$,

$$g_{xy}(\beta, \xi) = g_{xy}(\infty, \xi) + \beta^{-1} c_{xy}(\xi) + O(\beta^{-N}) \tag{4.19}$$

where $g_{xy}(\infty, \xi) \in \{\pm 1, \pm \varepsilon\}$, $\varepsilon = \alpha - 2$, and

$$c_{xy}(\xi) \in M = \left\{ t : t = \frac{1}{2} \ln \frac{m}{n},\, m, n \in \mathbb{N} \right\}, \qquad \mathbb{N} = \{1, 2, 3, \ldots\} \tag{4.20}$$

We prove Lemma 4.1 in several steps. First we prove some auxiliary results.

Lemma 4.2. If $\alpha \geqslant 2$, then for almost all ξ the equation

$$g_{xt} = f_\infty(\xi_x + g_{yx} + g_{zx}), \qquad \forall x \in V \tag{4.21}$$

has a unique solution $g = \{g_{xy}\}$.

Proof. We have the following properties

(1) If $\xi_x = \alpha$, then $g_{xt} \geqslant 0$;

(2) If $\xi_x = \alpha$ and $\max\{g_{yx}, g_{zx}\} \geqslant 0$ then $g_{xt} = 1$;

(3) If $\xi_x = \alpha$ and $\max\{\xi_y, \xi_z\} = \alpha$ then $g_{xt} = 1$;

The properties (1) and (2) are obvious from (4.21) and (3) follows from (1) and (2).

Denote by V_{xt} a half-tree with the root at x, which grows in the direction opposite to t. For a given configuration $\xi = \{\xi_x, x \in V\}$ consider the sets

$$\begin{aligned} &\Lambda_+(\xi) = \{v \in V_{xt} : \xi_v = \alpha, \max\{\xi_y, \xi_z\} = \alpha, S(v) = \{y, z\}\} \\ &\Lambda_-(\xi) = \Lambda_+(-\xi) \\ &\Lambda(\xi) = \Lambda_+(\xi) \cup \Lambda_-(\xi) \end{aligned} \tag{4.22}$$

For a given ξ, we say that there is no percolation by $V_{xt} \backslash \Lambda(\xi)$ if every path from x to ∞ contains a point from $\Lambda(\xi)$.

Lemma 4.3. For almost all ξ there is no percolation by $V_{xt} \backslash \Lambda(\xi)$.

Proof. Let $p_n^{\pm}$ be the probability of ξ's for which there is percolation from x to W_n under the condition that $\xi_x = \pm\alpha$, respectively. Then if we consider different possibilities for the field ξ_v at $v \in W_0 = \{x\}$ and $v \in W_1$, we obtain the recursive equations

$$\begin{cases} p_{n+1}^+ = \frac{1}{2} p_n^- - \frac{1}{4}(p_n^-)^2 \\ p_{n+1}^- = \frac{1}{2} p_n^+ - \frac{1}{4}(p_n^+)^2 \end{cases} \tag{4.23}$$

By symmetry $p_n^+ = p_n^-$, hence

$$p_{n+1}^+ \leqslant \tfrac{1}{2} p_n^+$$

which shows that $\lim_{n \to \infty} p_n^+ = 0$, so that with probability 1 there is no percolation from x to ∞. Lemma 4.3 is proved. ▮

End of the Proof of Lemma 4.2. For a given $\xi = \{\xi_x, x \in V\}$, define the set of points blocking the percolation, as

$$B(\xi) = \{v \in V_{xt} : v \in \Lambda(\xi) \text{ and } w \notin \Lambda(\xi) \quad \forall w \in \pi(x, v), w \neq v\} \tag{4.24}$$

where $\pi(x, v)$ is the path connecting x with v. By the property (3) above, $g_{vw}(\xi) = \pm 1$ if

$$v \in A_{\pm}(\xi),\ v \in S(w),\ w \in \pi(x, v),\ d(v, w) = 1$$

Hence the value $g_{vw}(\xi)$ is uniquely determined for $v \in B(\xi)$. In addition, if the blocking set $B(\xi)$ separates x from ∞, the value $g_{xt}(\xi)$ is uniquely determined by the values of $g_{vw}(\xi)$ on $v \in B(\xi)$ (by virtue of the recursive equation (4.21)). Since $B(\xi)$ does separate x from ∞ for almost all ξ, Lemma 4.2 is proved.

Assume that $\alpha = 2 + \varepsilon > 2$. Then the properties (1)–(3) can be strengthened as follows. Let $\beta = \infty$. Then

(1) If $\xi_x = \alpha$, then $g_{xt} \geqslant \varepsilon$

(2) If $\xi_x = \alpha$ and $\max\{g_{yx}, g_{zx}\} \geqslant \varepsilon$ then $(\xi_x + g_{yx} + g_{zx}) \geqslant 1 + 2\varepsilon$

(3) If $\xi_x = \alpha$ and $\max\{\xi_y, \xi_z\} = \alpha$ then $(\xi_x + g_{yx} + g_{zx}) \geqslant 1 + 2\varepsilon$

This allows us to prove that if $v \in A_{\pm}(\xi)$ then as $\beta \to \infty$,

$$g_{vw} = 1 + O(\beta^{-N}) \tag{4.25}$$

The proof is based on the asymptotic behavior of the function $f_\beta(t)$ as $\beta \to \infty$.

Lemma 4.4. As $\beta \to \infty$

$$f_\beta(t) = \begin{cases} t + O(\beta^{-N}) & \text{if } \ 0 \leqslant t < 1 \\ 1 + O(\beta^{-N}) & \text{if } \ t > 1 \end{cases} \tag{4.26}$$

and

$$f_\beta\left(1 + \frac{s}{\beta}\right) = 1 - \frac{\ln(1 + e^{-2s})}{2\beta} + O(\beta^{-N}) \tag{4.27}$$

We refer the reader to [BRZb] for the proof of Lemma 4.4. Lemma 4.4 shows that for $t \geqslant 1 + \varepsilon$ the function $f_\beta(t)$ is close to 1. We will call the region $t \geqslant 1 + \varepsilon$, the *plateau*.

Proof of Lemma 4.1. The property (3) above implies that if $\xi_v = \alpha$ and $\max\{\xi_y, \xi_z\} = \alpha$ where $S(v) = \{y, z\}$, then for sufficiently large β the value $(\xi_v + g_{yv} + g_{zv})$ is on the plateau, and, by virtue of Lemma 4.4, (4.25) holds. This proves (4.19) for $v \in A(\xi)$. Since the blocking set $B(\xi) \subset A(\xi)$, formula (4.25) holds for $v \in B(\xi)$. For almost all ξ the set $B(\xi)$ separates x

from ∞. Now, we can prove (4.19) for all v below $B(\xi)$ by induction moving down from $B(\xi)$ to x. So we assume that

$$\begin{cases} g_{yv}(\beta,\xi)=g_{yv}(\infty,\xi)+\beta^{-1}c_{yv}(\xi)+O(\beta^{-N}) \\ g_{yv}(\beta,\xi)=g_{zv}(\infty,\xi)+\beta^{-1}c_{zv}(\xi)+O(\beta^{-N}) \end{cases} \tag{4.28}$$

where $g_{yv}, g_{zv} \in \{\pm 1, \pm\varepsilon\}$ and $c_{yv}, c_{zv} \in M$. Then

$$g_{vt}(\beta,\xi)=f_\beta(\xi_v+g_{yv}(\beta,\xi)+g_{zv}(\beta,\xi)) \tag{4.29}$$

Consider three cases

(1) $|\xi_v+g_{yv}(\infty,\xi)+g_{zv}(\infty,\xi)|>1$

(2) $|\xi_v+g_{yv}(\infty,\xi)+g_{zv}(\infty,\xi)|<1$

(3) $|\xi_v+g_{yv}(\infty,\xi)+g_{zv}(\infty,\xi)|=1$

Then in case (1), Lemma 4.4 and (4.28) imply that

$$g_{vt}=\pm 1+O(\beta^{-N})$$

In case (2) we obtain that

$$\begin{aligned} g_{vt}(\beta,\xi) &= \xi_v+g_{yv}(\infty,\xi)+g_{zv}(\infty,\xi) \\ &\quad +\beta^{-1}c_{yz}(\xi)+\beta^{-1}c_{zv}(\xi)+O(\beta^{-N}) \\ &= g_{vt}(\infty,\xi)+\beta^{-1}(c_{yv}(\xi)+c_{zv}(\xi))+O(\beta^{-N}) \end{aligned}$$

This gives the asymptotics (4.19) with

$$c_{vt}=c_{yv}+c_{zv}\in M$$

(Observe that the set M in (4.20) is closed with respect to summation). In case (3) we similarly obtain that

$$g_{vt}(\beta,\xi)=g_{vt}(\infty,\xi)+\beta^{-1}c_{vt}(\xi)+O(\beta^{-N})$$

with

$$c_{vt}=-\text{sign}\, g_{vt}(\infty,\xi)\cdot\tfrac{1}{2}\ln(1+e^{-2(c_{yv}(\xi)+c_{zv}(\xi))})$$

In this case again $c_{vt}(\xi)\in M$, provided that $c_{yv}(\xi)$, $c_{zv}(\xi)\in M$. This induction proves Lemma 4.2 for $\alpha=2+\varepsilon>2$.

In the case $\alpha=2$ we have to use the following property (4) which follows from the ones (1)–(3).

Property (4): If $\xi_x = \alpha$ and either

(a) $\xi_y = \xi_z = \alpha$,

or

(b) $\xi_y = \alpha, \xi_z = -\alpha$ and $\max\{\xi_u, \xi_v\} = \alpha$ where $\{u, v\} = S(y)$,

or

(c) $\xi_y = -\alpha, \xi_z = \alpha$ and $\max\{\xi_u, \xi_v\} = \alpha$ where $\{u, v\} = S(z)$,

then $\xi_x + g_{yx} + g_{zx} \geqslant 2$. Indeed, in the case (a), $g_{xy}, g_{zx} \geqslant 0$ hence $\xi_x + g_{yx} + g_{zx} \geqslant 2$. In the case (b), $g_{yx} \geqslant 1$ by property (3) above, hence $\xi_x + g_{yx} + g_{zx} \geqslant 2 + 1 - 1 \geqslant 2$; the same arguments works for the case (c).

For a given ξ, we define the sets $\Lambda^0_\pm(\xi)$ of vertices $v \in V_{xt}$ for which the assumptions of Property (4) hold with respect to the configuration $\pm\xi$, respectively. Let $\Lambda^0(\xi) = \Lambda^0_+(\xi) \cup \Lambda^0_-(\xi)$.

We use the following lemma, which replaces Lemma 4.3.

Lemma 4.5. For almost all ξ there is no percolation by $V_{xt} \backslash \Lambda^0(\xi)$.

Proof. Let $p_n^\pm$ be probabilities of percolation from the root x to W_n under the condition that $\xi_x = \pm 2$. Then $p_n^+ = p_n^- = p_n$. Considering the different possibilities for ξ_y, ξ_z, ξ_u, and ξ_v, we obtain that

$$p_{n+1} \leqslant \tfrac{1}{2} p_n + \tfrac{1}{8}(p_n + 2p_{n-1}) = \tfrac{5}{8} p_n + \tfrac{1}{4} p_{n-1}$$

This is majorized by a sequence $\{b_n\}$ satisfying

$$b_{n+1} = \tfrac{5}{8} b_n + \tfrac{1}{4} b_{n-1}, \qquad b_0 = p_0, \quad b_1 = p_1$$

Two fundamental solutions for the last equation are $b_n = \lambda_{1,2}^n$ where $\lambda_{1,2}$ are to be found from the quadratic equation

$$\lambda^2 = \tfrac{5}{8}\lambda + \tfrac{1}{4} \qquad \text{or} \qquad 8\lambda^2 - 5\lambda - 2 = 0$$

This gives

$$\lambda_{1,2} = \frac{5 \pm \sqrt{89}}{16}$$

Since $|\lambda_{1,2}| < 1$, this implies that $\lim_{n\to\infty} b_n = 0$, hence $\lim_{n\to\infty} p_n = 0$. Lemma 4.5 is proved. ∎

The rest of the proof of Lemma 4.1 for $\alpha = 2$ is similar to the proof for $\alpha = 2 + \varepsilon > 2$ and we omit it.

Completion of the Proof of Theorem 4.1. Let us substitute the equation (4.19) into the formula for $\mu_{\beta,\xi}(\sigma_\Lambda)$:

$$\mu_{\beta,\xi}(\sigma_\Lambda) = Z^{-1} \exp[\beta H_0(\sigma_\Lambda) + H_1(\sigma_\Lambda) + O(\beta^{-N})] \tag{4.30}$$

where $\Lambda \subset V$, is a finite connected set,

$$H_0(\sigma_\Lambda) = \sum_{\substack{\langle x,y\rangle\\ x,y\in\Lambda}} \sigma_x\sigma_y + \sum_{x\in\Lambda} \xi_x\sigma_x + \sum_{\substack{\langle x,y\rangle\\ x\in\Lambda,\, y\in\Lambda^c}} g_{yx}(\infty,\xi)\,\sigma_x \tag{4.31}$$

and

$$H_1(\sigma_\Lambda) = \sum_{\substack{\langle x,y\rangle\\ x\in\Lambda,\, y\in\Lambda^c}} c_{xy}\sigma_x \tag{4.32}$$

Denote by $M_\xi(\Lambda) = \{\sigma_\Lambda^{(j)}\}$ the set of ground state configurations of the Hamiltonian $H_0(\sigma_\Lambda)$, i.e.,

$$\min_{\sigma_\Lambda} H_0(\sigma_\Lambda) = H_0(\sigma_\Lambda^{(j)}), \qquad \forall \sigma_\Lambda^{(j)} \in M_\xi(\Lambda)$$

Then the equation (4.30) implies that the limit,

$$\mu_{\infty,\xi}(\sigma_\Lambda) = \lim_{\beta\to\infty} \mu_{\beta,\xi}(\sigma_\Lambda)$$

exists and is concentrated on the ground state configurations. In addition, by (4.30)

$$\mu_{\infty,\xi}(\sigma_\Lambda) = Z^{-1} \exp H_1(\sigma_\Lambda) \tag{4.33}$$

In the case when $\Lambda = \{x\}$, (4.31) reduces to

$$H_0(\sigma_\Lambda) = \Big(\xi_x + \sum_{\langle x,y\rangle} g_{yx}(\infty,\xi)\Big)\sigma_x$$

hence if $x \in V_\pm(\xi)$, i.e.

$$\pm\Big(\xi_x + \sum_{\langle x,y\rangle} g_{yx}(\infty,\xi)\Big) > 0$$

then

$$\mu_{\infty,\xi}(\sigma_x = \pm 1) = 1, \qquad x \in V_\pm(\xi)$$

Let now $V_k(\xi)$ be a dipole polymer and let

$$B_k(\xi) = \{x \in V_k^c(\xi) : d(x, V_k(\xi)) = 1\}$$

be the boundary of $V_k(\xi)$. Then $B_k(\xi) \subset V_+ \cup V_-$. Therefore, for all ground states $\sigma_\Lambda^{(j)}$ on the set $\Lambda = V_k(\xi) \cup B_k(\xi)$ one gets $\sigma_x = \pm 1$ for $x \in B_k(\xi)$. By (4.33)

$$\mu_{\infty, \xi}(\sigma_\Lambda) = Z^{-1} \exp\left\{ \sum_{\substack{\langle x, y\rangle \\ x \in B_k(\xi),\, y \in \Lambda^c}} c_{xy}\sigma_x \right\}$$

This expression does not depend on $\{\sigma_x, x \in V_k(\xi)\}$, hence $\mu_{\infty, \xi}$ is a uniform measure, i.e.,

$$\mu_{\infty, \xi}(\sigma_\Lambda) = \frac{1}{|M_k(\xi)|}$$

where $|M_k(\xi)|$ is the number of ground states configurations of the Hamiltonian

$$H_{V_k(\xi)}(\sigma_\Lambda) = \sum_{\substack{\langle x, y\rangle \\ x, y \in V_k(\xi)}} \sigma_x\sigma_y + \sum_{x \in V_k(\xi)} \xi_x\sigma_x + \sum_{\substack{x \in V_k(\xi) \\ y \in B_k(\xi)}} \sigma_x\bar{\sigma}_y$$

where $\bar{\sigma}_y = \pm 1$ for $y \in V_\pm$, respectively. Let us show that the set $M_k(\xi)$ of ground states configurations coincides with the set of dipole configurations.

Since the Gibbs measure $\mu_{\beta, \xi}$ has the Markov property, the measure $\mu_{\infty, \xi}$ has it as well. This implies that

$$\mu_{\infty, \xi}(\sigma_{V_k(\xi)}) = \mu_{\infty, \xi}(\sigma_{y_0}) \prod_{\langle x, y\rangle \in L_k} \mu_{\infty, \xi}(\sigma_x \mid \sigma_y) \tag{4.34}$$

where L_k is the set of directed edges, which starts at some point $y_0 \in V_k(\xi)$ and which has the property that for every $x \in V_k(\xi)$, there exists a unique path by L_k from y_0 to x. Notice that

$$\mu_{\infty, \xi}(\sigma_x \mid \sigma_y) = \frac{\mu_{\infty, \xi}(\sigma_x, \sigma_y)}{\mu_{\infty, \xi}(\sigma_y)} \tag{4.35}$$

hence we deduce from (4.34) that $\mu_{\infty, \xi}(\sigma_{V_k(\xi)}) \neq 0$ for all dipole configurations and only for dipole configurations. Hence the set $M_k(\xi)$ of ground state configurations coincides with the set of dipole ground state configurations and Theorem 4.1 is proven.

5. RESIDUAL ENTROPY

In this section we will assume that $k=2$ and $2 \leqslant \alpha \leqslant 3$. By Theorem 2.4 (a) this ensures that the Gibbs state $\mu_{\beta,\xi}$ is unique for all $\beta<\infty$ and almost all ξ. We will derive some general formula for the entropy S_β of $\mu_{\beta,\xi}$. By S_β we understand the entropy on the "interior" spins (see, e.g., a discussion in [Ba]), and as well-known, on the Bethe lattice the entropy depends on the boundary conditions. We shall show that the entropy S_β is a "self-averaging" quantity, i.e., it is independent of ξ for *almost all* ξ. Then we shall calculate the residual entropy $S_\infty = \lim_{\beta\to\infty} S_\beta$ and show that $S_\infty > 0$.

Consider the stochastic recursive equation

$$g_x = f_\beta\left(\xi_x + \sum_{y\in S(x)} g_y\right) \tag{5.1}$$

It is understood as follows. Let $\{g_y, y\in S(x)\}$ be independent random variables with some distribution $\nu(dg)$, the same for all g_y's. Then we denote by $Q_\beta(\nu)(dg)$ the distribution of $f_\beta(\xi_x + \sum_{y\in S(x)} g_y)$, i.e., $Q_\beta(\nu)(dg)$ is the distribution of g_x in (5.1). A measure $\nu(dg)$ is called *invariant* with respect to Q_β if

$$\nu = Q_\beta(\nu)$$

Let $\mu^\pm_{\beta,\xi}$ be the Gibbs states with $(\pm)$-boundary conditions, respectively, and let $g^\pm(\beta,\xi) = \{g^\pm_x(\beta,\xi), x\in V\}$ be corresponding effective fields. Let $\nu^\pm_\beta(dg)$ be a probability distribution of $g^\pm_x(\beta,\xi)$. Observe that $\nu^\pm_\beta(dg)$ is independent of x.

From the definition of $\mu^\pm_{\beta,\xi}$ it follows that

$$\nu^\pm_\beta = \lim_{n\to\infty} Q^n_\beta(\nu^\pm) \tag{5.2}$$

where $\nu^\pm = \delta(g\pm 1)\,dg$. This implies that

$$\nu^\pm_\beta = Q_\beta(\nu^\pm_\beta) \tag{5.3}$$

i.e., $\nu^\pm_\beta$ are invariant measures. In addition,

$$\nu^-_\beta = S\nu^+_\beta, \qquad S: g \to -g \tag{5.4}$$

In the case when $\mu^+_{\beta,\xi} = \mu^-_{\beta,\xi}$ for almost all ξ, $\nu^+_\beta = \nu^-_\beta$ and it is symmetric.

Proposition 5.1. Assume that for a given $\beta<\infty$, $\mu^+_{\beta,\xi} = \mu^-_{\beta,\xi}$ for almost all ξ. Let $\nu = \nu^+ = \nu^-$. Then for all probability measures $\nu_0(dg)$ on $\mathbb{R}$,

$$\lim_{n\to\infty} Q^n_\beta(\nu_0) = \nu \tag{5.5}$$

Proof. Consider a half-tree V_0 with a root at x. From F.K.G.,

$$\langle \sigma_x \rangle_{\mu_n^-} \leqslant \langle \sigma_x \rangle_{\mu_n(\bar{\sigma})} \leqslant \langle \sigma_x \rangle_{\mu_n^+}$$

where $\mu_n^{\pm}$, $\mu_n(\bar{\sigma})$ are finite Gibbs distributions on V^0 with boundary conditions $\pm$ and $\bar{\sigma}$, respectively. Since

$$\langle \sigma_x \rangle_{\mu_n(\bar{\sigma})} = \tanh(\beta(\xi_x + h_x)) \tag{5.6}$$

this implies that

$$h_{xn}^- \leqslant h_{xn}(\bar{\sigma}) \leqslant h_{xn}^+$$

Since $g_x = f_\beta(\xi_x + h_x)$ this, in turn, implies that

$$g_{xn}^- \leqslant g_{xn}(\bar{\sigma}) \leqslant g_{xn}^+ \tag{5.7}$$

Consider random boundary conditions $\bar{\sigma} = \{\bar{\sigma}_x, x \in W_{n+1}\}$ where $\bar{\sigma}_x$ are independent random variables with the distribution ν_0. Then averaging with respect to $\bar{\sigma}$ we obtain that

$$g_{xn}^- \leqslant g_{xn}(\nu_0) \leqslant g_{xn}^+ \tag{5.8}$$

These inequalities hold for all ξ. Since $\mu_{\beta,\xi}^+ = \mu_{\beta,\xi}^-$ a.e. ξ, then taking in (5.8) $n \to \infty$, we obtain that

$$g_x^- = g_x(\nu_0) = g_x^+ \qquad \text{a.e. } \xi$$

and hence the distribution of $g_x(\nu_0) = \lim_{n\to\infty} g_{xn}(\nu_0)$ coincides with ν. Since the distribution of $g_{xn}(\nu_0)$ is nothing else than $Q^n(\nu_0)$, we obtain that $\nu = \lim_{n\to\infty} Q^n(\nu_0)$. Proposition 5.1 is proved. ∎

Now we turn to calculation of the entropy S_β. Consider the partition function of the $(\pm)$-state $\mu_{\beta,\xi}^{\pm}(\sigma_n)$,

$$Z_n^{\pm}(\beta, \xi) = \sum_{\sigma} \exp\left(\beta \sum_{\langle x, y\rangle \in L_n} \sigma_x \sigma_y + \beta \sum_{x \in V_n} \xi_x \sigma_x + \beta \sum_{x \in W_n} h_x^{\pm}(\xi)\, \sigma_x\right) \tag{5.9}$$

The free energy (density) is defined as

$$F(\beta) = \lim_{n\to\infty} -\frac{1}{\beta\,|V_n|} \ln Z_n^{\pm}(\beta, \xi) = \lim_{n\to\infty} -\frac{1}{\beta 3(2^n)} \ln Z_n^{\pm}(\beta, \xi) \tag{5.10}$$

Observe that $|W_n| = 3(2^{n-1})$ and $|V_n| = 3(2^n) - 2$.

Theorem 5.1. The free energy exists for all ξ, it is independent of ξ, and it is the same for (+)-state and (−)-state. The free energy is given by the formula

$$F(\beta) = -\sum_{\xi=\pm\alpha} \int d_\beta\left(\xi + \sum_{y \in S(x)} g_y\right) \prod_{y \in S(x)} \nu_\beta^{\pm}(dg_y) \tag{5.11}$$

where $\nu_\beta^{\pm}(dg)$ is the invariant measure of the stochastic equation (5.1) and

$$d_\beta(x) = (2\beta)^{-1} \ln(4 \cosh \beta(x+1) \cosh \beta(x-1)) \tag{5.12}$$

Proof. For the sake of definiteness, let us consider (+)-state. From formula (1.7), we obtain the recursive equation

$$Z_n^+(\beta, \xi) = \exp\left(\beta \sum_{x \in W_n} d_\beta(\xi_x + h_x^+(\xi))\right) Z_{n-1}^+(\beta, \xi) \tag{5.13}$$

This gives that

$$F^+(\beta, \xi) = \lim_{n \to \infty} -\frac{1}{3(2^n)} \sum_{k=0}^{n} \sum_{x \in W_{n-k}} d_\beta(\xi_x + h_x^+(\xi))$$

Observe that $|d_\beta(\xi_x + h_x^+(\xi))| \leqslant C_\beta$ for all ξ_x, $h_x^+(\xi)$, hence

$$\frac{1}{3(2^n)} \sum_{k=\ell+1}^{n} \sum_{x \in W_{n-k}} d_\beta(\xi_x + h_x^+(\xi)) \leqslant \frac{C_\beta}{3(2^n)} \left(\sum_{k=\ell+1}^{n} 3(2^{n-k-1})\right) \leqslant C_\beta \cdot 2^{-\ell}$$

Therefore,

$$F^+(\beta, \xi) = \lim_{\ell \to \infty} \lim_{n \to \infty} -\frac{1}{3(2^n)} \sum_{k=0}^{\ell} \sum_{x \in W_{n-k}} d_\beta(\xi_x + h_x^+(\xi))$$

Since the random variables $\xi_x + h_x^+(\xi)$, $x \in W_n$, are independent, and the distribution of $\xi_x + h_x^+(\xi)$ is the same for all $x \in V$, we obtain, by the law of large numbers, that for a fixed k, for almost all ξ,

$$\lim_{n \to \infty} -\frac{1}{3(2^{n-k-1})} \sum_{x \in W_{n-k}} d_\beta(\xi_x + h_x^+(\xi)) = C = -\sum_{\xi=\pm\alpha} \int d_\beta(\xi + h)\, \nu^+(dh) \tag{5.14}$$

where $\nu^+(dh)$ is the distribution of h. This implies that for almost all ξ,

$$F^+(\beta, \xi) = \lim_{\ell \to \infty} \sum_{k=0}^{\ell} \frac{1}{2^{k+1}} C = C$$

Since $h_x^+ = \sum_{y \in S(x)} g_y^+$, formula (5.11) follows from (5.14). In addition, (5.4) implies that $F^-(\beta) = F^+(\beta)$. Theorem 5.1 is proved. ▮

In the case when the Gibbs state is unique, formula (5.11) reduces to

$$F(\beta) = -\sum_{\xi = \pm\alpha} \int d_\beta\left(\xi + \sum_{y \in S(x)} g_y\right) \prod_{y \in S(x)} \nu_\beta(dg_y) \tag{5.15}$$

where $\nu_\beta = \nu_\beta^+ = \nu_\beta^-$.

Differentiability of the Free Energy. The free energy is a function of β and α. The contraction argument that we used to prove the uniqueness of the Gibbs state (see Theorems 2.1, 2.2, and 2.4) allows us to prove also that the free energy is infinitely differentiable in β and α in the indicated regions of uniqueness in the $\beta - \alpha$ plane. Indeed, let us consider for the sake of definiteness the differentiability in β.

Differentiation of recursive equation (3.2) in β gives a recursive equation on $\partial_\beta g_x$:

$$\partial_\beta g_x = (\partial_\beta f_\beta)\left(\xi_x + \sum_{y \in S(x)} g_y\right) + f'_\beta\left(\xi_x + \sum_{y \in S(x)} g_y\right) \sum_{y \in S(x)} \partial_\beta g_y \tag{5.16}$$

where $f'_\beta(t) = df_\beta(t)/dt$. This equation implies that if we have two solutions of (3.2), g_x^1 and g_x^2, then

$$|\partial_\beta g_x^1 - \partial_\beta g_x^2| \leqslant C[1 + \max_{y \in S(x)} |\partial_\beta g_y^1|] \max_{y \in S(x)} |g_y^1 - g_y^2| + kf'_\beta(c) \max_{y \in S(x)} |\partial_\beta g_y^1 - \partial_\beta g_y^2| \tag{5.17}$$

where

$$c = \xi_x + \sum_{y \in S(x)} g_y^1$$

Assume that we know (as in Theorems 2.1, 2.2) that $0 < kf'_\beta(c) < q < 1$. Let $g_x^1 = g_{xm}(\xi)$ and $g_x^2 = g_{xn}(\xi)$ be solutions of recursive equation (3.2) in the volumes V_m and V_n, respectively, with, say, $(+)$-boundary conditions. Then

1. there exist some constants $C_0, c_0 > 0$ such that $|g_x^1 - g_x^2| < C_0 e^{-c_0 l}$ where $l = \min\{m, n\}$ (see the proof of Theorems 2.1 and 2.2 above);

2. (5.16) implies that $\sup_n |\partial_\beta g_{xn}| < \infty$;

3. (5.17) implies that there exist some constants $C_1, c_1 > 0$ such that $|\partial_\beta g_x^1 - \partial_\beta g_x^2| < C_1 e^{-c_1 l}$, $l = \min\{m, n\}$.

Hence the Cauchy criterion holds for $\partial_\beta g_{xn}(\xi)$ which proves that $g_x(\xi) = \lim_{n\to\infty} g_{xn}(\xi)$ is differentiable in β for all ξ. Similarly, if we write ξ_x as $\xi_x = \alpha\eta_x$ where $\eta_x = \pm 1$ then we can prove that g_x is differentiable in α for all $\eta = \{\eta_x, x\in V\}$. Higher order differentiation of (3.2) in β and α allows us to prove in the same way that g_x is an infinitely differentiable function with respect to β and α for all η.

If, like in Theorem 2.4a, we have a contraction only for the mathematical expectations of g_x with respect to ξ (or with respect to η where $\xi = \alpha\eta$), then the above argument allows us to prove the differentiability in β and α of the mathematical expectation $\mathbb{E}_\eta A(g_x, x\in\Lambda)$, where Λ is an arbitrary finite subset of V and $A(g_x, x\in\Lambda)$ is an arbitrary smooth function. Observe that the free energy in (5.11) is a mathematical expectation of this type, hence it is infinitely differentiable in β and α in the regions indicated in Theorems 2.1, 2.2, and 2.4.

Evaluation of the Residual Entropy. The entropy S_β can be obtained from (5.11) as

$$S_\beta = -\frac{dF}{dT}(\beta), \qquad \beta = T^{-1}$$

The residual entropy at $T=0$ is then

$$S_\infty = -\lim_{\beta\to\infty} \frac{F(\beta) - F(\infty)}{(1/\beta)} \tag{5.18}$$

where $F(\infty) = \lim_{\beta\to\infty} F(\beta)$. By Lemma 4.1, $\lim_{\beta\to\infty} g_x(\beta,\xi) = g_x(\infty,\xi)$ takes values in $\{\pm 1, \pm\varepsilon\}$. Thus the distribution $\nu_\infty = \lim_{\beta\to\infty} \nu_\beta(dg)$ has the form

$$\nu_\infty = [p\delta(g+1) + q\delta(g+\varepsilon) + q\delta(g-\varepsilon) + p\delta(g-1)]\, dg, \qquad \varepsilon = \alpha - 2 \tag{5.19}$$

The weights p, q satisfy $p + q = \frac{1}{2}$ and they are determined from the fixed point stochastic equation

$$\nu = Q_\infty(\nu) \tag{5.20}$$

Assume that $0 < \varepsilon < 1$. Then (5.20) reduces to the equations

$$q = \tfrac{1}{2}\cdot p^2, \qquad p + q = \tfrac{1}{2}$$

This gives

$$p=\sqrt{2}-1, \qquad q=\tfrac{3}{2}-\sqrt{2} \tag{5.21}$$

To derive the residual entropy S_∞ from (5.18) we need the linear term in the asymptotics of $\nu_\beta(dg)$ as $\beta^{-1}\to 0$. By Lemma 4.1, $\forall N>1$,

$$g_x(\beta,\xi)=g_x(\infty,\xi)+c_x(\xi)\,\beta^{-1}+O(\beta^{-N})$$

where $g_x(\infty,\xi)$ takes values in the set $\{\pm 1, \pm\varepsilon\}$ and $c_x(\xi)$ takes values in the set

$$M=\left\{\frac{\ln(m/n)}{2}, m, n=1, 2,...\right\}$$

Let $x_0=0$, x_1, x_2,... be an enumeration of the points in M. Then we obtain that at $\beta\to\infty$, $\nu_\beta(dg)$ is approximated by the distribution

$$\nu_\beta^{\text{asymp}}(dg)=\left[\sum_{a=\pm 1,\,\pm\varepsilon}\;\sum_{j=0}^{\infty} p_{a,j}\delta\left(g-a-\frac{x_j}{\beta}\right)\right]dg, \qquad x_j\in M \tag{5.22}$$

in the sense that for any smooth test function $\varphi(g)$,

$$\int \varphi(g)[\nu_\beta(dg)-\nu_\beta^{\text{asymp}}(dg)]=O(\beta^{-N}) \tag{5.23}$$

The weights $p_{a,j}$ are found from the fixed point equation $\nu=Q_\beta(\nu)$. They satisfy the equations

$$\sum_{j=0}^{\infty} p_{1,j}=p+O(\beta^{-N}), \qquad \sum_{j=0}^{\infty} p_{\varepsilon,j}=q+O(\beta^{-N}) \tag{5.24}$$

Formula (5.11) for the free energy can be written as

$$F(\beta)=-\int d_\beta(s)\; W_\beta(ds) \tag{5.25}$$

where $W_\beta(ds)$ is the probability distribution of

$$\sigma_x=\xi_x+\sum_{y\in S(x)} g_y \tag{5.26}$$

From (5.22), (5.23) we obtain that $W_\beta(ds)$ is approximated by the distribution

$$W_\beta^{\mathrm{asymp}}(ds)=\left[\sum_{a\in A}\sum_{j=0}^{\infty}w_{a,j}\delta\left(s-a-\frac{x_j}{\beta}\right)\right]dg,\qquad x_j\in M \tag{5.27}$$

where

$$A=\{\xi+g_y+g_z \mid \xi=\pm(2+\varepsilon):g_y,\, g_z=\pm 1,\, \pm\varepsilon\}$$

and $w_{a,j}$ are some weights expressed in terms of $p_{a,j}$. The approximation means that for every test function $\varphi(s)$,

$$\int \varphi(s)[W_\beta(ds)-W_\beta^{\mathrm{asymp}}(ds)]=O(\beta^{-N})$$

The function

$$d_\beta(s)=\frac{1}{2\beta}\ln[2\cosh(2\beta)+2\cosh(2\beta s)]$$

is even and it has the following asymptotics as $\beta\to\infty$:

$$d_\beta(s)=\begin{cases}1+O(\beta^{-N}) & \text{if}\quad 0<s<1\\ s+O(\beta^{-N}) & \text{if}\quad s>1\end{cases} \tag{5.28}$$

and

$$d_\beta\left(1+\frac{s}{\beta}\right)=1+\frac{1}{2\beta}\ln(1+e^{2s})+O(\beta^{-N}) \tag{5.29}$$

Combining these asymptotic formulas with (5.27), we derive from (5.25) that

$$\begin{aligned}F(\beta)=&-2\sum_{0<a<1}\sum_{j=0}^{\infty}w_{a,j}-2\sum_{a>1}\sum_{j=0}^{\infty}\left(a+\frac{x_j}{\beta}\right)w_{a,j}\\&-2\sum_{j=0}^{\infty}\left[1+\frac{1}{2\beta}\ln(1+e^{2x_j})\right]w_{1,j}+O(\beta^{-N})\end{aligned} \tag{5.30}$$

hence

$$F(\beta)=F(\infty)-\frac{1}{\beta}\left[2\sum_{a>1}\sum_{j=0}^{\infty}x_jw_{a,j}+\sum_{j=0}^{\infty}\ln(1+e^{2x_j})\,w_{1,j}\right]+O(\beta^{-N}) \tag{5.31}$$

From (5.18) we obtain the residual entropy at $T=0$ as

$$S_\infty = 2 \sum_{a>1} \sum_{j=0}^{\infty} x_j w_{a,j} + \sum_{j=0}^{\infty} \ln(1+e^{2x_j})\, w_{1,j} \tag{5.32}$$

This is an exact formula. Observe that S_∞ does not depend on $\alpha = 2+\varepsilon$, $0<\varepsilon<1$.

We do not have an analytic expression for $w_{a,j}$ but some estimates and numerics shows that the weight $w_{1,0}$ is noticeably larger than the other weights. If we keep in (5.32) only the term $w_{1,0}$ then (5.32) reduces to the Bruinsma approximation (see [Br]),

$$S_\infty^{(0)} = w_1 \ln 2 \tag{5.33}$$

From (5.27) we find $w_1 = pq$, hence

$$S_\infty^{(0)} = pq \ln 2 = \frac{5\sqrt{2}-7}{2} \ln 2 \approx 0.035 \ln 2, \qquad 2<\alpha<3 \tag{5.34}$$

Formula (5.32) remains valid for $\alpha=2$ and $\alpha=3$, with some different weights $w_{a,j}$. Approximation (5.33) can also be extended to $\alpha=2$ and $\alpha=3$. It gives (cf. [Br]):

$$S_\infty^{(0)} = \begin{cases} \frac{1}{8} \ln 2 & \text{if} \quad \alpha = 3 \\ \frac{1}{16} \ln 2 & \text{if} \quad \alpha = 2 \end{cases}$$

which gives the values of $S_\infty^{(0)}$ at the spikes $\alpha = 2, 3$ higher than the value (5.34) on the plateau $2<\alpha<3$.

6. PROOF OF THEOREMS 2.5 AND 2.6

Proof of Theorem 2.5 from Theorem 2.6. Note that by symmetry

$$\mathbb{E}_\xi \int \sigma_x \mu^+_{\beta,\xi}(\sigma) = -\mathbb{E}_\xi \int \sigma_x \mu^-_{\beta,\xi}(\sigma) = M(\beta, \alpha)$$

hence by Theorem 2.6

$$\mathbb{E}_\xi \left(\int \sigma_x \mu^+_{\beta,\xi}(\sigma) - \int \sigma_x \mu^-_{\beta,\xi}(\sigma) \right) = 2M(\beta,\alpha) > 0$$

In addition by F.K.G.,

$$\int \sigma_x \mu^+_{\beta,\xi}(\sigma) - \int \sigma_x \mu^-_{\beta,\xi}(\sigma) \geqslant 0$$

We would like to prove that for almost all ξ we actually have a strict inequality, at least for one x. Define the random variable

$$F(\xi) = \sum_{x \in V} a(x) \left(\int \sigma_x \mu^+_{\beta,\xi}(\sigma) - \int \sigma_x \mu^-_{\beta,\xi}(\sigma) \right)$$

where $a(x) > 0$ are arbitrary numbers such that

$$\sum_{x \in V} a(x) < \infty$$

Then $F(\xi) \geqslant 0$ and $\mathbb{E}_\xi F(\xi) > 0$. Define

$$A = \{\xi : F(\xi) > 0\}$$

Then $\Pr A > 0$, because otherwise $\mathbb{E}_\xi F(\xi) = 0$. Let T: $V \to V$ be a shift of the Bethe lattice. Notice that

$$\int \sigma_{Tx} \mu^+_{\beta, T\xi}(\sigma) = \int \sigma_x \mu^+_{\beta,\xi}(\sigma)$$

Hence, if $F(\xi) > 0$, then $F(T\xi) > 0$. Therefore, $A = TA$. Since $\{\xi_x\}$ are independent, the shift T is ergodic and consequently $\Pr A = 0$ or 1. Since $\Pr A > 0$, actually, $\Pr A = 1$. Hence, for almost all ξ

$$\int \sigma_x \mu^+_{\beta,\xi}(\sigma) - \int \sigma_x \mu^-_{\beta,\xi}(\sigma) > 0$$

for at least one x. This proves that $\mu^+_{\beta,\xi} \neq \mu^-_{\beta,\xi}$ for almost all ξ and ends the proof of Theorem 2.5.

Proof of Theorem 2.6. Let g_x satisfy the basic equation

$$g_x = f_\beta(\xi_x + g_y + g_z) \tag{6.1}$$

Assuming that g_y and g_z are independent and have the same distribution $\rho(dg)$, the equation (6.1) determines a distribution of g_x, which we denote by $Q\rho(dg)$. We are interested in the behavior of $\rho_k = Q^k \rho_0$ as $k \to \infty$, assuming that

$$\rho_0(dg) = \delta(g-1)\,dg \tag{6.2}$$

which corresponds to $(+)$-boundary conditions, see (1.2) and (3.3).

The key point is some inductive assumptions on ρ_k which hold for ρ_0 and which are reproducible when we pass from ρ_k to ρ_{k+1}. To formulate these inductive assumptions we need some definitions. Let $\alpha = H^{\mathrm{F}}(T) + \varepsilon$. We will assume that $\varepsilon > 0$ is sufficiently small, so that it satisfies some conditions formulated below. Let

$$f_{\pm}(g) = f_{\beta}(\pm\alpha + 2g) \tag{6.3}$$

and let $a > 0$ be the point where

$$f'_{-}(a) = 1; \qquad f'_{-}(g) < 1 \quad \forall g > a \tag{6.4}$$

(see Fig. 7). Observe that

$$f_{-}(a) = a - \varepsilon$$

Indeed, let $f(t) = f_{\beta}(2t)$ and let $g_0 > 0$ be a solution of the equation $f'(g_0) = 1$. Then

$$H^{\mathrm{F}}(T) = f(g_0) - g_0, \qquad a = \alpha + g_0, \qquad f_{-}(a) = f(g_0)$$

From here,

$$a - f_{-}(a) = \alpha + g_0 - f(g_0) = \alpha - H^{\mathrm{F}}(T) = \varepsilon$$

which was stated.

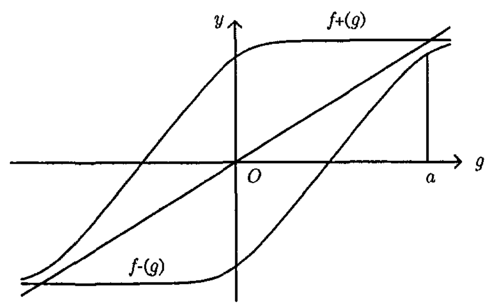

Fig. 7. A narrow corridor of order $O(\varepsilon)$ in the vicinity of $g = a$ is responsible for appearance of a long intermittent trajectory near a and finally for asymmetric distribution of effective field, cf. Fig. 3. The same phenomenon one has in the vicinity of $g = -a$.

Near a, there is a narrow corridor of width of order $O(\varepsilon)$, between the diagonal $y = g$ and the graph of equation $y = f_-(g)$. This implies that we have a long intermittent trajectory $\{g_n = f_-(g_{n-1})\}$ near a.

Consider some points b, c, d such that $a < b < c < d$ and such that when $\varepsilon \to 0$,

$$\varepsilon \ll c - b \ll b - a = \frac{d-c}{2} \ll 1 \tag{6.5}$$

The notation $f \ll g$ as $\varepsilon \to 0$ means that $\lim_{\varepsilon \to 0} f/g = 0$. Define a sequence $b_0 > b_1 > \cdots > b_N$ by the recursive equation

$$b_{n-1} = f_-^{-1}(b_n), \qquad N \geqslant n \geqslant 1, \qquad b_N = a \tag{6.6}$$

We assume that $b = b_0$, i.e.,

$$a = b_N, \qquad b = b_0 \tag{6.7}$$

and we choose

$$N = [\,|\ln \varepsilon|\,]$$

Define then $b_{N+1}, b_{N+2},\ldots$ by the equation

$$b_n = b_{n-1} - \varepsilon, \qquad n \geqslant N+1 \tag{6.8}$$

Let

$$\begin{aligned} p^+(k) &= \int_d^\infty \rho_k(dg) \\ p_0(k) &= \int_b^c \rho_k(dg) \\ p_n(k) &= \int_{b_n}^{b_{n-1}} \rho_k(dg), \qquad n \geqslant 1 \end{aligned} \tag{6.9}$$

To avoid technical difficulties we will assume that $\rho_k(\{d\}) = \rho_k(\{c\}) = \rho_k(\{b_n\}) = 0$ for all n.

Inductive assumption I_k

(i) $p^+(k) \geqslant 0.499$;

(ii) $p_0(k) \leqslant 0.13$;

(iii) $p_n(k) \leqslant \frac{1}{8} \cdot 2^{-n}$, $n = 1, 2,\ldots.$

Main Lemma. There exists $\varepsilon(T)>0$, which depends continuously on $0<T<T_c$, such that for $0<\varepsilon<\varepsilon(T)$, $I_k \Rightarrow I_{k+1}$, $k=0, 1, 2,\ldots$.

Proof. Assume that (i)–(iii) hold for ρ_k and prove that (i)–(iii) hold for ρ_{k+1}.

Proof of (i). Observe that by (6.5), $0<d-a\ll 1$. This implies that

$$f_+(g)>d \qquad \text{if} \quad g>a$$

(see Fig. 7). In addition,

$$\Pr\{g_y \leqslant a\} \leqslant \sum_{n=N+1}^{\infty} \tfrac{1}{8}\cdot 2^{-n} = \tfrac{1}{8}\cdot 2^{-N}$$

Therefore

$$\begin{aligned}\Pr\{g_x \geqslant d\} &\geqslant \Pr\{\xi_x=\alpha\}\cdot \Pr\{g_y \geqslant a\}\cdot \Pr\{g_z \geqslant a\}\\ &\geqslant \tfrac{1}{2}\cdot(1-2^{-N})^2 \geqslant 0.499\end{aligned}$$

if N is sufficiently large. This proves (i).

Proof of (ii). Assume that $g_x \in [b, c]$. Then by (6.1) two cases are possible:

Case 1. $\xi_x=\alpha$ and

$$\frac{g_y+g_z}{2} \in f_+^{-1}([b, c])$$

and

Case 2. $\xi_x=-\alpha$ and

$$\frac{g_y+g_z}{2} \in f_-^{-1}([b, c])$$

Let us estimate probabilities of these two cases.

Case 1. From Fig. 7 it is clear that if $g\in[b, c]$ then $f_+^{-1}(g)<a$, hence either $g_y<a$ or $g_z<a$. The probability of this possibility is evaluated by

$$\delta = 2\cdot \Pr\{\xi_x=\alpha\}\cdot \Pr\{g_y<a\} \leqslant 2^{-N} \ll 1$$

Case 2. In this case,

$$\frac{g_y + g_z}{2} \in f_-^{-1}([b, c]) = [b_{-1}, c_{-1}] \tag{6.10}$$

where $b_{-1} = f_+^{-1}(b)$, $c_{-1} = f_+^{-1}(c)$. From Fig. 7 it is clear that $|b - b_{-1}|$, $|c - c_{-1}| \leqslant C\varepsilon$, hence

$$0 < c_{-1} - b_{-1} \ll d - c$$

and $c_{-1} < d$, so that

$$\frac{g_y + g_z}{2} < d$$

Consider two cases for g_y, g_z.

Case (a): $g_y, g_z < d$. By (i),

$$\Pr\{g_y < d\} \leqslant 0.501$$

hence the probability of this case is estimated by

$$\delta_1 = \Pr\{\xi_x = -\alpha\} \cdot \Pr\{g_y < d\} \cdot \Pr\{g_z < d\} \leqslant \tfrac{1}{2} \cdot 0.501^2 < 0.126 \tag{6.11}$$

Case (b): either $g_y \geqslant d$ or $g_z \geqslant d$. Let, say, $g_y \geqslant d$. Then by (6.5) and (6.10),

$$g_z < a$$

(use that $d - c = 2(b - a)$ and $0 < c_{-1} - b \ll d - c$), hence the probability of this case is estimated by

$$\delta_2 = 2 \cdot \Pr\{\xi_x = -\alpha\} \cdot \Pr\{g_z < a\} < 2^{-N} \ll 1$$

Thus, $p_0(k+1) \leqslant \delta + \delta_1 + \delta_2 < 0.13$. This proves (ii).

Let us prove (iii) for $p_n(k+1)$. First we consider $n = 1, 2$, then $2 < n \leqslant N$, and finally $n > N$.

Proof of (iii) for $p_1(k+1)$. If $g_x \in [b_1, b_0]$ and $\xi_x = \alpha$, then

$$\frac{g_y + g_z}{2} \in f_+^{-1}([b_1, b_0]) < a$$

hence either $g_y < a$ or $g_z < a$, and the probability of this case is estimated by $\delta = 2^{-N} \ll 1$. Assume that $\xi_x = -\alpha$. Then

$$\frac{g_y + g_z}{2} \in f_-^{-1}([b_1, b_0]) = [b_0, b_{-1}], \qquad b_{-1} = f_-^{-1}(b_0)$$

Consider two cases

Case (a), $g_y, g_z \geqslant b_0 = b$. Then $g_y, g_z < c$, because otherwise $(g_y + g_z)/2 > b_{-1}$ (use that $0 < b_{-1} - b_0 < C\varepsilon \ll c - b$). Hence the probability of this case is estimated by

$$\delta_1 = \Pr\{\xi_x = -\alpha\} \cdot (\Pr\{g_y \in [b, c]\})^2 \leqslant \tfrac{1}{2} \cdot 0.13^2$$

Case (b), either $g_y < b_0$ or $g_z < b_0$. Let, say, $g_y < b_0$. Consider two subcases,

Subcase (b_1), $b_0 < g_z < c$. The probability of this subcase is estimated by

$$\delta_2 = 2 \cdot \Pr\{\xi_x = -\alpha\} \cdot \Pr\{g_y < b_0\} \cdot \Pr\{g_z \in [b, c]\}$$

where the factor 2 comes from the possibility to exchange g_y and g_z. Since

$$\Pr\{g_y < b_0\} = \sum_{n=1}^{\infty} \Pr\{g_y \in (b_n, b_{n-1}]\} \leqslant \tfrac{1}{8}$$

we obtain that

$$\delta_2 \leqslant \tfrac{1}{8} \cdot 0.13$$

Subcase (b_2), $g_z > c$. Then

$$g_y = 2 \cdot \frac{g_y + g_z}{2} - g_z < 2b_{-1} - c$$

Since $c - b \gg \varepsilon$, this implies that

$$g_y < b_{N_0}, \qquad N_0 \gg 1$$

and therefore,

$$\Pr\{g_y < 2b_{-1} - c\} \leqslant 2^{-N_0}$$

Thus, the probability of this subcase is estimated by $\delta_3 = 2^{N_0} \ll 1$.

Combining all cases and subcases we obtain that

$$p_1(k+1) < \delta + \delta_1 + \delta_2 + \delta_3 \leqslant \tfrac{1}{2} \cdot 0.13^2 + \tfrac{1}{8} \cdot 0.13 + \delta + \delta_3 \leqslant 0.026$$

This proves that $p_1(k+1) < \frac{1}{16}$, hence I_{k+1}(iii) holds for $n = 1$.

Proof of $p_2(k+1) < \frac{1}{32}$. As before, the case $\xi_x = \alpha$ has a negligibly small probability $\delta \ll 1$. Let $\xi_x = -\alpha$. Then

$$\frac{g_y + g_z}{2} \in f_-^{-1}([b_2, b_1]) = [b_1, b_0]$$

Consider two cases.

Case (a), $g_y, g_z \geqslant b_1$. Observe that either $g_y \leqslant b_0$ or $g_z \leqslant b_0$, hence the probability of this case is estimated by

$$\begin{aligned} \delta_1 &= 2 \cdot \Pr\{\xi_x = -\alpha\} \cdot \Pr\{g_y \in [b_1, b_0]\} \cdot \Pr\{g_z \in [b_1, c]\} \\ &\leqslant \tfrac{1}{16} \cdot (\tfrac{1}{16} + 0.13) \leqslant \tfrac{1}{32} \cdot 0.4 \end{aligned}$$

Case (b), either $g_y < b_1$ or $g_z < b_1$. Let, say, $g_y < b_1$. Consider two subcases,

Subcase (b_1), $g_z < c$. The probability of this subcase is estimated by

$$\delta_2 = 2 \Pr\{\xi_x = -\alpha\} \cdot \Pr\{g_y < b_1\} \cdot \Pr\{g_z \in [b_1, c]\}$$

Observe that by I_k,

$$\Pr\{g_y \in [b_n, b_{n-1}]\} \leqslant \tfrac{1}{8} \cdot 2^{-n}, \qquad \Pr\{g_y \leqslant b_n\} \leqslant \tfrac{1}{8} \cdot 2^{-n}$$

Since

$$\Pr\{g_y < b_1\} \leqslant \tfrac{1}{16}$$

we get that

$$\delta_2 \leqslant \tfrac{1}{16} \cdot (\tfrac{1}{16} + 0.13) + 2^{-N} < \tfrac{1}{32} \cdot 0.4$$

Subcase (b_2), $g_z \geqslant c$. Then $g_y \leqslant b_{N_0}$ and the probability of this subcase is estimated by $\delta_3 \ll 1$. Thus,

$$p_2(k+1) < \delta + \delta_1 + \delta_2 + \delta_3 < \tfrac{1}{32}$$

This proves (iii) for $n = 2$.

Estimate of $p_n(k+1)$ for $3 \leqslant n \leqslant N$. As before, the case $\xi_x = \alpha$ has a negligibly small probability δ such that $2^N \delta \ll 1$. Let $\xi_x = -\alpha$. Then

$$\frac{g_y + g_z}{2} \in f_-^{-1}([b_n, b_{n-1}]) = [b_{n-1}, b_{n-2}]$$

Consider two cases.

Case (a), $g_y, g_z \geqslant b_{n-1}$. Observe that either $g_y \leqslant b_{n-2}$ or $g_z \leqslant b_{n-2}$. Let, say $g_y \leqslant b_{n-2}$. Then $g_z \in [b_{n-1}, b_{n-3}]$, because

$$b_{n-2} - b_{n-3} > b_{n-1} - b_{n-2}$$

(use that $f'_-(g) < 1$ for $g > a$). Hence the probability of this case is estimated by

$$\begin{aligned}\delta_1 &= 2 \cdot \Pr\{\xi_x = -a\} \cdot \Pr\{g_y \in [b_{n-1}, b_{n-2}]\} \cdot \Pr\{g_z \in [b_{n-1}, b_{n-3}]\} \\ &\leqslant \tfrac{1}{8} \cdot 2^{-(n-1)} \cdot \tfrac{1}{8}(2^{-(n-1)} + 2^{-(n-2)}) = \tfrac{3}{16} \cdot 2^{-2n}\end{aligned}$$

Case (b), either $g_y < b_{n-1}$ or $g_z < b_{n-1}$. The probability of this subcase is estimated by

$$\begin{aligned}\delta_2 &= \Pr\{g_y < b_{n-1}\} \cdot \Pr\{g_z \in [b_{n-2}, c]\} \leqslant \tfrac{1}{8} \cdot 2^{-(n-1)} \cdot (\tfrac{1}{8} + 0.13) \\ &= \tfrac{1}{8} \cdot 2^{-n} \cdot 0.51\end{aligned}$$

Thus,

$$p_n(k+1) \leqslant \delta + \delta_1 + \delta_2 \leqslant \delta + \tfrac{1}{8} \cdot 2^{-n} \cdot (\tfrac{3}{2} \cdot 2^{-n} + 0.51) < \tfrac{1}{8} \cdot 2^{-n}$$

This finishes proof of (iii) for $n \leqslant N$.

Proof of (iii) for $n > N$. Let us consider $\xi_x = -\alpha$. Then

$$\frac{g_y + g_z}{2} \in f_-^{-1}([b_n, b_{n-1}])$$

Since $f'_-(g) < 2 \tanh \beta$ and $f(a) = a - \varepsilon$, we obtain that

$$f_-^{-1}(b_{n-1}) > b_m, \qquad m = \frac{n}{2 \tanh \beta} - C_0$$

where C_0 does not depend on ε. This implies that

$$f_-^{-1}([b_n, b_{n-1}]) \subset [b_j, b_{j-\ell}]$$

where

$$j \geqslant \frac{n}{2 \tanh \beta} - C_0$$

and ℓ is a number that does not depend on ε. Hence

$$\frac{g_y + g_z}{2} \in [b_j, b_{j-\ell}]$$

Let us estimate

$$\delta = \Pr\left\{\frac{g_y + g_z}{2} \in [b_j, b_{j-\ell}]\right\}$$

If $(g_y + g_z)/2 \in ([b_j, b_{j-\ell}]$ then either $g_y \leqslant b_{j-\ell}$ or $g_z \leqslant b_{j-\ell}$. Let, say, $g_y \leqslant b_{j-\ell}$. Assume that $g_y \in [b_{m+1}, b_m]$. Then $g_z \in [b_{p+1}, b_p]$, with

$$2j - m - C_1 < p < 2j - m + C_1$$

where C_1 does not depend on ε. Therefore,

$$\begin{aligned} \Pr\left\{\frac{g_y + g_z}{2} \in [b_j, b_{j-\ell}]\right\} &\leqslant C_2 \sum_{m=j}^{2j} 2^{-m} 2^{-(2j-m)} \leqslant 2C_2 j 2^{-2j} \\ &\leqslant C_3 n 2^{-2n/(2 \tanh \beta)} = C_3 n 2^{-n/\tanh \beta} \end{aligned}$$

Since $\tanh \beta < 1$, this implies that

$$\Pr\left\{\frac{g_y + g_z}{2} \in [b_j, b_{j-\ell}]\right\} \leqslant \frac{1}{10} \cdot 2^{-n}$$

and $p_n(k+1) \leqslant \frac{1}{8} \cdot 2^{-n}$. This finishes the proof of the inductive assumption I_{k+1}. Main Lemma is proven. ▮

Completion of the Proof of Theorem 2.6. The one-point distribution of the $(+)$-state is given by

$$\mu^+_{\beta, \xi}(\sigma_x) = Z^{-1} \exp \sigma_x \left(\xi_x + \sum_{y : d(x, y) = 1} g_y^+(\xi)\right)$$

The effective field $g_y^+(\xi)$ has the distribution

$$\nu^+ = \lim_{k \to \infty} Q^k \nu_0, \qquad \nu_0(dg) = \delta(g - 1)\, dg$$

By Main Lemma, ν^+ satisfies the inductive assumption I_k. Observe that

$$\mathbb{E}_\xi\langle\sigma_x\rangle = Z^{-1} \sum_{\xi_x=\pm\alpha} \prod_{y:d(x,y)=1} \int \nu^+(dg_y) \times \sum_{\sigma_x=\pm 1} \sigma_x \exp \sigma_x \left[\xi_x + \sum_{y:d(x,y)=1} g_y^+(\xi)\right]$$

By I_k,

$$\lim_{\varepsilon\to 0^+} \int_0^\infty \nu^+(dg) = 1, \qquad \varepsilon = \alpha - H^{\mathrm{F}}(T)$$

which implies that for sufficiently small $\varepsilon > 0$,

$$\mathbb{E}_\xi\langle\sigma_{x_0}\rangle > 0$$

This finishes the proof of Theorem 2.6. ∎

On the Discontinuous Change of the Support of the Invariant Measure $\nu_\beta^+(dg)$ at $a = \boldsymbol{H}^{\mathrm{F}}(T)$. Let $T < T_c$. Then for $0 < \alpha \leqslant H^{\mathrm{F}}(T)$ the support of the limiting measure

$$\nu_\beta^+(dg) = \lim_{k\to\infty} Q^k \nu_0(dg), \qquad \nu_0(dg) = \delta(g-1)\,dg$$

lies in the interval

$$\operatorname{supp} \nu_\beta^+ \subset [M^+(\beta, -\alpha), M^+(\beta, \alpha)] \tag{6.12}$$

where $t = M^+(\beta, \pm\alpha) > 0$ is the largest among three solutions of the fixed point equation

$$f_\beta(\pm\alpha + 2t) = t$$

Indeed, by the F.K.G. inequality, for all realizations ξ of the random external field,

$$M^+(\beta, -\alpha) = g_x^+(\{-\alpha\}; \beta) \leqslant g_x^+(\xi; \beta) \leqslant g_x^+(\{\alpha\}; \beta) = M^+(\beta, \alpha)$$

hence the support of the distribution of $g_x^+(\xi; \beta)$, which is $\nu_\beta^+(dg)$, lies in the interval $[M^+(\beta, -\alpha), M^+(\beta, \alpha)]$ on the positive half-axis, which was stated.

For $\alpha > H^F(T)$ the fixed point equation $f_\beta(-\alpha + 2t) = t$ has a unique solution $M^-(\beta, -\alpha) < 0$. We claim that for every $\gamma > M^-(\beta, -\alpha)$,

$$\int_{-\infty}^{\gamma} \nu_\beta^+(dg) > 0 \tag{6.13}$$

Indeed, let $N = N(\gamma)$ be such a number that for all $t_0 \leqslant 1$,

$$f_-^N(t_0) < \gamma$$

where $f_-(t) = f_\beta(-\alpha + 2t)$ and f_-^N means the Nth iteration of the map $f_-: t \to f_-(t)$. There is a positive probability $p(N) > 0$ that $\xi_y = -\alpha$ for all y in the ball of radius $N+1$ centered at x_0. In this case the recursive equation $g_x = f_\beta(\xi_x + g_y + g_z)$ implies that

$$g_x = f_\beta(-\alpha + g_y + g_z) \leqslant f_-(t), \qquad t = \max\{g_y, g_z\}$$

for all x in the ball of radius N, hence

$$g_{x_0} \leqslant f_-^N(1) < \gamma$$

with probability at least $p(N) > 0$, which was stated.

The relations (6.12) and (6.13) show that at $\alpha = H^F(T)$ the support of the invariant measure $\nu_\beta^+(dg)$ changes discontinuously. Since the free energy $F^+(\beta, \alpha)$ is expressed as an average with respect to a finite product of the measures $\nu_\beta^+(dg)$ (see formula (5.11) above), we conjecture that $F(\beta, \alpha)$ is nonanalytic in α at $\alpha = H^F(T)$ but we cannot prove it rigorously.

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